N63-20891

NASA TECHNICAL NOTE

NASA TN D-2006

NASA TN D-2006

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Prepared under Grant No. NsG-198-62 *by*
CASE INSTITUTE OF TECHNOLOGY
Cleveland, Ohio
for

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • SEPTEMBER 1963

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ABSTRACT

Consideration is given to the performance characteristics of an AC magnetohydrodynamic power generator. A rotating magnetic field is imposed on the vortex flow of an electrically conducting fluid, which is injected tangentially into an annulus formed by two nonconducting concentric cylinders and two non-conducting end plates. A perturbation technique is used to determine the two dimensional velocity and three dimensional electromagnetic field and current distributions. Finally, the generated power, the ohmic losses, the effective power and the electrical efficiency of the converter system are calculated.

*This investigation was submitted in partial fulfillment of the requirement for the Degree of Doctor of Philosophy.

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INTRODUCTION

MHD GENERATOR CONCEPT

The direct conversion of thermal energy into electrical power by magnetohydrodynamic (MHD) power generators is attracting a great deal of attention presently. The high temperatures associated with such energy sources as nuclear reactors and solar furnaces prevent the direct application of the conventional energy converters such as gas or steam turbines. Furthermore, the development of new converter devices is highly motivated by the fact that the thermal efficiency of the converter system increases with the temperature at which the power conversion takes place.

For the MHD Generators the following general concept has been developed:

Thermal energy is transferred from a high temperature source to a working fluid increasing its internal and kinetic energies. The fluid is made electrically conducting either by thermal ionization or by seeding conducting plasma into it or by both processes simultaneously, and it is passed through a magnetic field. As the result of interaction of the magnetic field with the conducting fluid an electromotive force is induced which in turn produces an electric current distribution in the conducting medium.

If the magnetic field and the flow pattern are so chosen that the induced currents are steady in time DC power generation results and the energy can be extracted from the fluid through electrodes in the walls of the flow duct.

If the induced currents are periodic functions of time, AC power is generated and it can be extracted either through electrodes or, by utilizing the magnetic flux linkage, through the exciting field coils. In the last case the MHD generator operates on the principles of a conventional induction generator.

As the result of the energy extraction and some irreversible processes inherent to the generator operation, the total pressure of the fluid decreases, and conventional converter devices may become applicable at the lower energy (temperature) levels.

REVIEW OF PREVIOUS INVESTIGATIONS

The concept of MHD power generator is not new; it was conceived first by Faraday, who proposed to utilize the motion of ocean-water in the earth's magnetic field for power generation [1]*.

Recently, with the advent of high temperature energy sources, the idea of direct conversion of flow-enthalpy into electrical energy has begun to undergo a vigorous investigation.

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* Numbers in brackets indicate References.

From the beginning, most of the attention was focused on DC generators because of their relative simplicity. In addition, DC generators have some operational advantage over the comparative AC devices, especially in the case of large scale power generation.

Among others, R. J. Rosa, A. R. Kantrowitz and T. R. Brogan have investigated the general feasibility and performance characteristics of MHD DC generators (see [2] to [5].) An experimental generator operating with plasma produced by an arc wind tunnel was built and operated by them at AVCO Laboratory in Everett, Massachusetts. G. W. Sutton of General Electric Co. [6], [7] presented a detailed analysis of a channel-type MHD DC generator in 1959. The analysis is restricted to the discussion of the one-dimensional channel-type motion of an inviscid conducting fluid in the presence of a normal magnetic field.

Similar investigations were performed by S. Way of Westinghouse Research Laboratories [8], [9], who used combustion-product gases as the working medium in a channel-type linear generator. The performance characteristics obtained by him were about half of those theoretically predicted.

A different model for DC power generation was proposed by J. McCune of Aeronautical Research Associates of Princeton [10], who conducted both theoretical and experimental investigations on a vortex flow formed between two concentric cylinders placed in a steady axial magnetic field. The end-plate effects were neglected in this analysis. The results obtained by McCune indicate the general feasibility of the device for power generation.

DC-generation systems have some inherent disadvantages. First of all, the use of large scale DC generators within the existing commercial power systems would require the installation of a number of large, expensive DC-AC alternators. Furthermore, the DC generator cycle itself possesses a number of undesirable characteristics such as the electron absorption by the electrodes at the operational temperatures. The necessity of effective electrode cooling raises some additional problems [11].

In view of these disadvantages more recently attention was given to the idea of electrodeless MHD generators producing directly AC currents and utilizing magnetic flux linkage instead of electrodes.

A relatively small amount of work has been done so far on the development of AC-MHD generators. This is probably due to the fact that the induction generators have a number of unfavorable operational characteristics, too; some of them are considered to be serious enough to cause doubts about the general feasibility of such devices for large scale power generation. For example, it can be shown that the power generated by any MHD-device is in general proportional to the conductivity of the working fluid, the square of its velocity and to the square of the magnetic field strength interacting with the fluid:

$$P_g \propto \sigma V^2 B^2$$

On the other hand, the reactive power (P_r) to be supplied for maintaining an alternating magnetic field for AC power generation is proportional to $\omega B^2/\mu$; where μ is the magnetic permeability of the conductor and $\omega/2\pi$ is the frequency of the exciting field. (One may note here that the reactive power inherent to AC generators has no counterpart in DC-devices.) Thus the ratio of the power produced to the reactive power supplied is given for an AC generator as

$$\frac{P_g}{P_r} \propto \frac{R_m}{\omega^*} \quad ;$$

where $R_m \equiv \sigma\mu VL$ is defined as the magnetic Reynolds number and $\omega^* = \omega L/V$ is a dimensionless frequency parameter and L is a characteristic length in the system. Hence, for $\omega^* = O(1)$ (a realistic value for commercial generator systems) the ratio of the power produced to the reactive power is proportional to the magnitude of the magnetic Reynolds number. Practical values of R_m in MHD generator application range from 0.01 to 0.1. Hence, the reactive power supplied to the system is unduly high, as compared to the amount of power generated within the system. The large reactive power requires the use of costly capacitive equipment on one hand, and it causes substantial reactive losses in the field coils on the other hand. These losses can partially be compensated by installing capacitor-banks in the circuit. The increase of capital cost due to such compensation, however, is not in proportion to the net power gained.

Recently, in connection with the appearance of super-conductors with associated high critical fields and high current carrying capacities, and of cryogenic capacitors with high quality factors, the interest in MHD-AC generators has been renewed. A limited number of papers have been published on the subject in more recent times.

I. Bernstein of the Forrestal Research Center at Princeton University, and others [12], investigated the slug motion (= constant velocity) of a conducting medium between two infinite plates and, in particular, its interaction with a time dependent magnetic field travelling parallel to the direction of motion of the conductor. The electrical efficiency corresponding to max. power output was found to be 1/2.

H. Woodson of Massachusetts Institute of Technology [13] analyzed the interaction of a plasma slug (or a sequence of plasma slugs) travelling downstream in a shock tube with solenoidal magnetic field. He obtained some basic requirements for AC-generator action as applied to MHD generator devices.

A comprehensive description of the basic principles of MHD induction generators was given in the lecture series on Engineering MHD offered by MIT in June 1961 [14].

SUBJECT AND SCOPE OF THE PRESENT WORK

In the following, an induction type MHD generator will be analyzed whose operation is based on the interaction of a "rotating" magnetic field with a vortex flow of a conducting fluid rotating in the plane of the magnetic flux lines. The choice of this system is suggested by its relative compactness and the limited work done previously on the analysis of rotating MHD fields.

The "rotating" magnetic field is really the result of superposition of two pulsating fields with a phase shift between them:

$$B_x = B_o \sin \omega t \quad ; \quad B_y = B_o \cos \omega t \quad .^*$$

Superimposing B_x and B_y, it appears to a stationary observer that a field vector of constant magnitude $\vec{B}_o$ "rotates" in the x-y plane with a frequency equal to $\omega/2\pi$.

The same induction field can be described in cylindrical-polar coordinate system as

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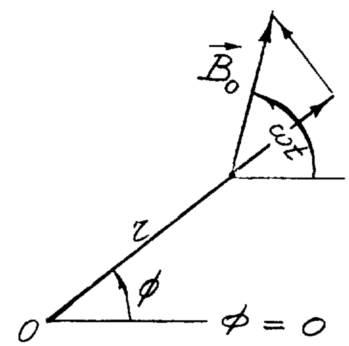
* All symbols are defined in the "List of Symbols".

$$B_o = \hat{r}\, B_o \cos(\omega t - \phi) + \hat{\phi}\, B_o \sin(\omega t - \phi)$$



Such a rotating field is produced, for example, by polyphase windings used in conventional induction generators.

The generator chamber is formed by two nonconducting coaxial cylindrical walls and two parallel non-conducting end plates (Fig. 1).

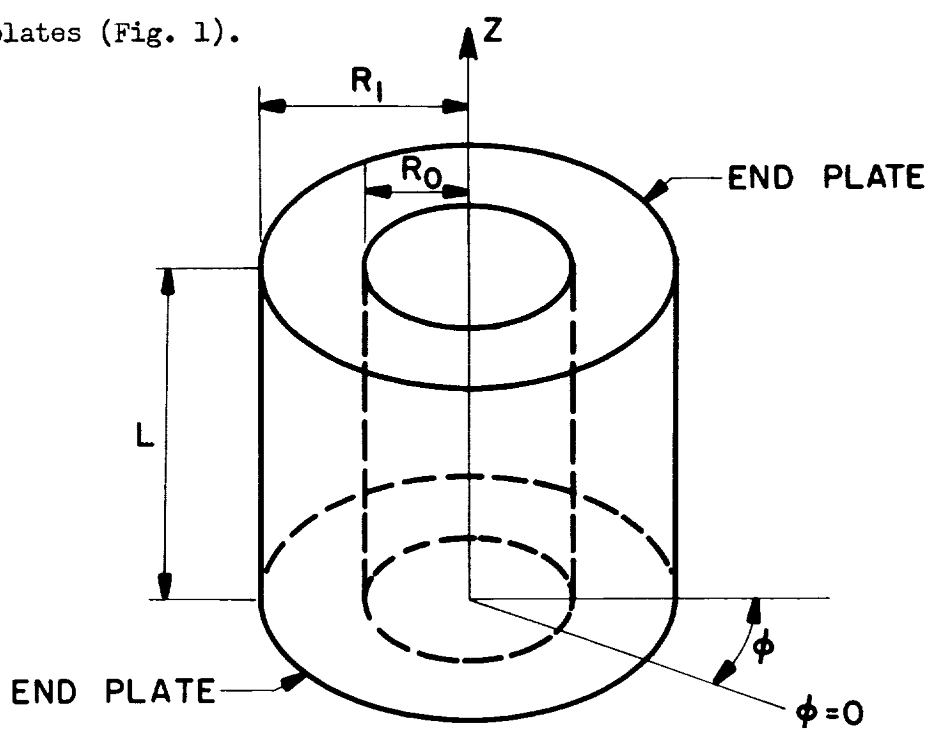


Fig. 1.

The working fluid is injected tangentially at the outer radius of the cylindrical annulus and it leaves the chamber in the radial direction through the ports on the inner cylinder (Fig. 2).

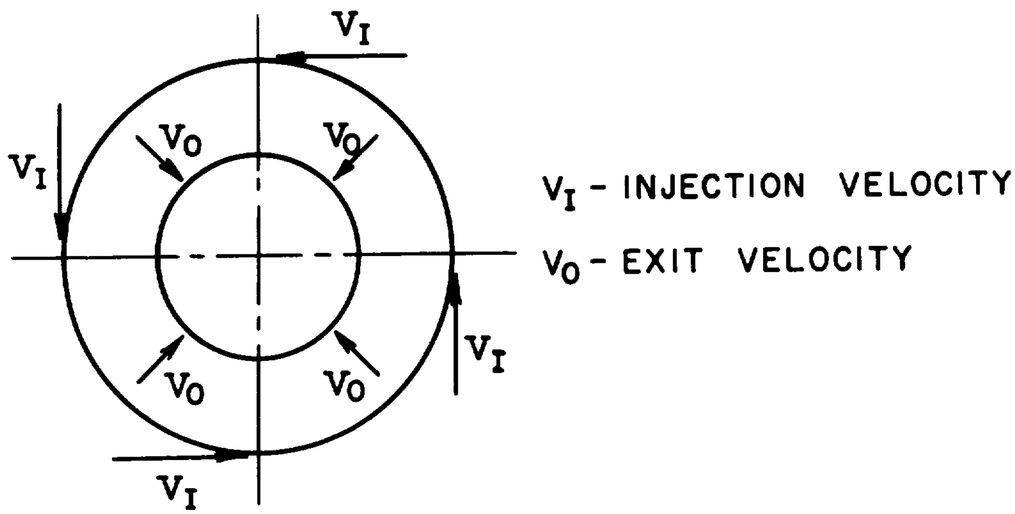


Fig. 2.

Rotation of the fluid faster than the magnetic field will induce an alternating current distribution in the "fluid rotor" which will have its own magnetic field. The induced field interacts directly with the exciting polyphase windings, thus the energy can be extracted from the system through the same field coils which induce the primary field. In this sense the device acts as a conventional AC induction generator.

For the configuration described above, the three-dimensional electromagnetic fields, the velocity and the current distributions will be determined by approximate techniques.

Following this, the power generated and the ohmic losses will be calculated. This in turn will furnish the necessary information about the performance characteristics and the internal electrical efficiency of the generator cycle.

Finally, an attempt will be made to determine the optimal parameters of the system corresponding to maximum operating efficiency.

ASSUMPTIONS AND LIMITATIONS

The complexity of the three-dimensional problem described in the previous section necessitates the introduction of a series of simplifying assumptions in the analytic treatment.

Since the intensity of interaction between the hydrodynamic and electromagnetic fields is determined by the magnitude of the magnetic Reynolds number, R_m, and $R_m \ll 1$ in the case of MHD generators due to the low electrical conductivity of the available working media, the applicability and usefulness of perturbation technique with R_m as characteristic parameter becomes apparent.

Thus the analysis will be restricted to cases when the assumption $R_m \ll 1$ holds and series expansion in positive powers of R_m will be applied to the various field quantities.

The present work is limited to the computation of the zeroth and first order terms where the zeroth order field distributions correspond to the complete absence of interaction between the magnetic and velocity fields. The higher order terms introduce corrections to the zeroth order quantities.

As will be shown later, for acceptable convergence of the higher order terms , restriction must be made on an additional electromagnetic parameter; the magnetic pressure coefficient defined as $S \equiv B_o^2/\mu_o \rho V^2$ must be of unit-order (ρ is the fluid density and V is a characteristic velocity in the hydrodynamic field).

The above restrictions imposed on R_m and S assure adequate accuracy in the numerical computations even if only the zeroth and first order terms are considered, as was shown by Rossow [15].

Although the electrical conductivity of a plasma obtained by thermal ionization is a function of the temperature which changes as the power is being extracted, it will be assumed that the generation cycle can be described in terms of an average -- or effective electrical conductivity which can be considered as a constant during the process. Such an approximation yields reasonable results if the temperature change during a cycle is not very large. In MHD generators the power extracted is approximately equal to (or at least of the same order as) the ohmic losses within the working substance, hence substantial temperature changes may not be expected in general.

The Hall effects are completely neglected throughout the present analysis. Thus the electrical conductivity will be treated as a scalar quantity. This in turn implies sufficiently high pressures for the working medium so that the product of the electron Larmour frequency with the electron collision time will have a value small compared to unity (see Ref. [2]). This is a reasonable approximation for the case of MHD generator systems.

Complete axial symmetry is assumed for the injection and exhaust systems only. Such systems can practically be obtained, for example, by injecting the plasma through the wall of a rotating porous cylinder and letting it to leave the generator chamber through a stationary porous cylinder, the last being coaxial with the first.

The present analysis will be restricted to the discussion of laminar flow regimes. Such an approach was shown to be realistic by J. McCune (see Ref. [10]), whose results indicate also the general feasibility of laminar vortex flow patterns in MHD generator application.

The presence of the hydrodynamic boundary layers on the end plates will be completely neglected. Although, due to this simplification, an exact solution for the two-dimensional zeroth order hydrodynamic (Navier-Stokes) equations becomes available in the (r , ϕ) plane, only the inviscid (potential) solution will be applied in the subsequent determination of the electromagnetic field distributions. This is necessitated by the difficulties introduced in the subsequent mathematical development by the presence of terms containing the hydrodynamic Reynolds number as a parameter.

One has to make clear distinction at this point between the end-plate effects connected with the hydrodynamic fields and those connected with the electromagnetic field distributions.

The neglect of the hydrodynamic boundary layers on the end-plates is justified by the fact that the primary fluid motion is a plane motion directed parallel to the end plates. Thus the disturbance introduced on the flow field by the presence of the end-plates can be localized to the vicinity of those plates.

Since both the primary fluid motion and the imposed magnetic induction field are localized to the (r, ϕ) plane, the induced electromotive force and also the primary current flow will be directed normal to the end-plates. Thus the neglect of the influence of the non-conducting end-plates on the zeroth order electromagnetic field and current distributions would be highly misleading.

Therefore, the three-dimensional zeroth order electromagnetic field and current distributions will be determined with full account taken of the presence of the non-conducting end-plates. The only approximations involved here will be those connected with the use of the potential velocity distribution.

Further approximations are required, however, for the determination of the first order field and current distributions due to the time-dependent, asymmetric character of the differential equations involved. Solutions to these equations are obtained omitting completely not only the viscous effects but also the end-plate effects on the electromagnetic field distributions.

From the above discussion it becomes apparent that the hydrodynamic boundary layer effects are completely omitted both in the zeroth and the first order solutions. Such an approximation is justified for large hydrodynamic Reynolds' numbers or for large scale generators because the influence of the boundary layer on the various field quantities is essentially a surface effect, and as the relative thickness of the boundary layer decreases (as by increasing the size of the generator or decreasing the viscosity of the working fluid or by both applied simultaneously), its influence on the total energy output becomes less significant.

It should be mentioned here, however, that the boundary layer losses in AC - MHD generators are more pronounced than in comparative DC-devices.

In DC-generators the emf within the boundary layer still has the same sign as that in the potential flow, but it is reduced in magnitude. Therefore, the boundary layer in DC-generators forms simply a leakage path between the electrodes to shunt the load.

In AC-generators, on the other hand, power is generated in that part of the flow region only, where the velocity of the conducting medium exceeds the propagation velocity of the magnetic field (this is the case of "negative slip", where the slip velocity is defined in electrical machinery as the difference between the propagation velocity of the magnetic field and the velocity of the conductor). In flow regions where the fluid lags the propagating magnetic field, power is consumed to speed up the fluid to the synchronous velocity and instead of power generation, pumping action takes place. Since within the boundary layers the velocity is reduced to zero at stationary walls, at least a part of the boundary layer will always act not only as a leakage path for eddy currents, but also as a power consuming region. As has been indicated, these losses are not considered herein.

Since the power supplied to the field coils for maintaining the applied magnetic field is much larger than the power transmitted to the load (as has been shown: the ratio of the maximum power output to the power input is proportional to the magnitude of the magnetic Reynolds number), the distortion of the applied magnetic field by the characteristics of the load circuit will be completely neglected.

Finally, in the process of the following analysis, the working fluid will be assumed to be incompressible. This implies some limitations on the magnitude of the injection velocity.

PART A. FIELD DISTRIBUTION

1. MATHEMATICAL FORMULATION OF THE PROBLEM

1. Basic Equations

The laminar motion of an incompressible electrically conducting fluid in the presence of a magnetic field can be described in terms of the following equations:

Mass conservation:

$$\nabla \cdot \vec{v} = 0 \text{*} \tag{1.1.1}$$

Momentum conservation:

$$\rho \frac{D\vec{V}}{Dt} = -\nabla p + \vec{I} \times \vec{B} + \vec{F} + \nu \nabla^2 \vec{v} \tag{1.1.2}$$

Maxwell's relations:

$$\nabla \times \vec{B} = \mu \vec{I} + \mu \varepsilon \frac{\partial \vec{E}}{\partial t} \tag{1.1.3}$$

$$\nabla \cdot \vec{B} = 0 \tag{1.1.4}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{1.1.5}$$

$$\nabla \cdot \vec{E} = \rho_e / \varepsilon \tag{1.1.6}$$

The generalized Ohm's Law, after neglecting the Hall currents, can be written as

---

*For the notation, see the List of Symbols.

$$\vec{I} = \sigma(\vec{E} + \vec{V} \times \vec{B}) \tag{1.1.7}$$

Since the divergence of the current density must vanish,

$$\nabla \cdot \vec{I} = 0 \tag{1.1.8}$$

The energy equation will be introduced and discussed in a later section.

2. MHD Approximation

One frequently encounters magnetohydrodynamic problems dealing with fluids and plasmas of low electrical conductivity and negligible net charge accumulation in the bulk of the fluid. In such cases the applied electric and magnetic fields cause merely a relative motion of the charged particles in an almost neutralized state. Thus the possibility of charge accumulation is usually excluded except in the vicinity of non-conducting boundaries, and the bulk of the fluid can be considered as an electrically neutral medium. The electrical charge density ρ_e is assumed to be zero in most of the flow region.

The magnetic permeability of the fluid can be approximated usually by using the permeability of a vacuum space.

Furthermore, the displacement and magnetization currents can also be neglected usually.

With the above approximation the Maxwell equations can be rewritten as

$$\nabla \times \vec{B} = \mu_0 \vec{I} \tag{1.2.1}$$

$$\text{with} \quad \nabla \cdot \vec{B} = 0 \quad ; \tag{1.2.2}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{1.2.3}$$

$$\text{with} \quad \nabla \cdot \vec{E} = 0 \quad \text{or} \quad \nabla \cdot \vec{I} = 0 \tag{1.2.4}$$

$$\text{where} \quad \vec{I} = \sigma(\vec{E} + \vec{V} \times \vec{B}) \tag{1.2.5}$$

Equations (1.2.1) and (1.2.2) define the magnetic field; the electric field can be determined subsequently using equations (1.2.3) and (1.2.4). The $\vec{E}$ term in (1.2.5) contains really two parts: an induced electric field given by (1.2.3) and measured in a coordinate system where the magnetic induction field has a non-vanishing time derivative, and a static electric field determined by the boundary conditions. The static field can be described usually in terms of a potential function.

Careful consideration should be given to the use and limitations of equation (1.2.1). Although in small magnetic Reynolds number applications equation (1.2.1) is often neglected and the current distribution is determined by using equation (1.2.5) exclusively, it should be clearly understood that equation $\nabla \times \vec{B} = \mu_0 \vec{I}$ has a very well defined meaning even if the induced magnetic field is much smaller than the applied induction field.

The role of equation (1.2.1) in small magnetic Reynolds number application can be seen from the following argument.

If the interaction of an applied magnetic field, $\vec{B}_o$, with a conducting fluid is considered, then the term $\mu_o^{-1} \nabla \times \vec{B}_o$ gives the current distribution $\vec{J}_o$ which induces the "outside" (= applied) magnetic field. Hence

$$\nabla \times \vec{B}_o = \mu_o \vec{J}_o$$

relates the applied magnetic field with the current distribution maintaining the field $\vec{B}_o$.

Assuming now boundary conditions such that $\vec{E} \equiv 0$ in the flow region, the current distribution in the moving fluid is given by

$$\vec{I} = \sigma (\vec{V} \times \vec{B}) \quad , \qquad \text{where}$$

$$\vec{B} = \vec{B}_o + \vec{b}_i \quad ;$$

$\vec{b}_i$ being the magnetic induction field generated by the induced currents. Equation (1.2.1) yields now:

$$\nabla \times (\vec{B}_o + \vec{b}_i) = \mu_o \vec{J}_{total} \quad , \text{ where}$$

$$\vec{J}_{total} = \vec{J}_o + \vec{I} \; .$$

But $\nabla \times \vec{B}_o = \mu_o \vec{J}_o \quad ,$

hence $\nabla \times \vec{b}_i = \mu_o \vec{I} \quad .$

If the magnetic Reynolds number is sufficiently low, then $\vec{b}_i \ll \vec{B}_o$ and Ohm's Law can be rewritten as $\vec{I} \simeq \sigma(\vec{V} \times \vec{B}_o)$, but the equation $\nabla \times \vec{b}_i = \mu_o \vec{I}$ remains unchanged and valid, since the condition $\vec{b}_i \ll \vec{B}_o$ does not imply any restriction on the derivatives of $\vec{b}_i$.

From the above considerations follows that the equation $\nabla \times \vec{b}_i = \mu_o \vec{I}$ can always be used. For that part of the space where $\vec{J}_o = 0$ the above equation can be replaced by $\nabla \times \vec{B} = \mu_o \vec{I}$.

3. Boundary Conditions

The assumed hydrodynamic boundary conditions are as follows:

The fluid of given azimuthal velocity V_I and pressure p_I is injected tangentially into the vortex chamber at the outer cylinder with an approximate axisymmetrical velocity distribution. (This condition can be obtained, for example, by injecting the fluid through a rotating porous cylinder.) The fluid leaves the chamber through radial ports on the inner cylinder (a stationary porous inner cylinder would correspond to the idealized example given above). The radial velocity at the outer cylinder is defined by the continuity condition. Since the end-plate effects are neglected, no boundary conditions will be ascribed in the z-direction. Using cylindrical polar coordinates the above considerations imply that

$$v_\phi = V_I \quad \text{at} \quad r = R_I \tag{1.3.1}$$

$$v_\phi = 0 \quad \text{at} \quad r = R_o \tag{1.3.2}$$

$$v_r = Q_o/A_o \quad \text{at} \quad r = R_o \tag{1.3.3}$$

where Q_o is the total volume flow and A_o is the area of the exit ports at $r = R_o$. In addition, it is assumed that

$$v \neq v\,(z) \tag{1.3.4}$$

In general, the following boundary conditions can be applied to the electromagnetic fields:

The tangential component of the electric field is continuous at an interface (such as the wall of the generator chamber); $E_{t2} = E_{t1}$ (1.3.5)

The change of the normal component of the electrical field at an interface is equal to the surface charge density:

$$E_{2n} - E_{1n} = - \sigma_e / \epsilon_o \qquad (1.3.6)$$

The normal component of the magnetic field intensity is continuous at the interface: $H_{n2} - H_{n1} = 0$ (1.3.7)

The change of the tangential component of the magnetic field intensity at the interface is equal to the surface current density distributed there:

$$\hat{n} \times (\vec{H}_2 - \vec{H}_1) = \vec{K} \qquad (1.3.8)$$

Furthermore, considering a conducting medium bounded by finite nonconducting boundaries, the normal component of the current must vanish at the nonconducting envelope. Thus the total electric field must have a vanishing normal component at the boundaries.

4. Nondimensionalization of the Equations

The characteristic or reference values for the physical quantities described by the basic equations will be chosen as follows:

The actual width of the flow duct $R_I - R_o = \Delta R$ shall be used as reference length.

Since the performance characteristics of a conventional induction generator are strongly influenced by the magnitude of the so-called slip velocity (defined in electrical engineering as the excess velocity of the propagating magnetic field over the velocity of the conductor), we shall choose for a reference velocity a quantity numerically equal to the negative slip velocity at the outer radius:

$$V_I - \omega R_I = \Delta V$$

($\omega/2\pi$ is the frequency of the applied induction field.)

The reference quantity for the magnetic field will be the applied induction field strength B_o ; the electric field will be nondimensionalized by the product ($\Delta V\, B_o$).

The following dimensionless quantities can be introduced now:

$$x_i^* = \frac{x_i}{R_I - R_o} = \frac{x_i}{\Delta R} ;$$

$$\vec{v}^* = \frac{\vec{v}}{V_I - \omega R_I} = \frac{\vec{v}}{\Delta V} ;$$

$$t^* = \frac{t\Delta V}{\Delta R} ; \qquad \omega^* = \frac{\omega \Delta R}{\Delta V} ; \tag{1.4.1}$$

$$p^* = \frac{p}{\rho V_I^2} \cdot \left(\frac{V_I}{\Delta V}\right)^2 ; \quad \vec{B}^* = \frac{\vec{B}}{B_o} ;$$

$$\vec{E}^* = \frac{\vec{E}}{(\Delta V)B_o} ; \quad \vec{I}^* = \frac{\vec{I}}{\sigma(\Delta V)B_o} .$$

With the aid of the nondimensional quantities presented here, the basic equations can be written in the following form:

$$\nabla^{*} \cdot \vec{V}^{*} = 0 \tag{1.4.2}$$

$$\frac{D\vec{V}^{*}}{Dt^{*}} + \nabla^{*} p^{*} = N(\vec{E}^{*} + \vec{V}^{*} \times \vec{B}^{*}) \times \vec{B}^{*} + \frac{1}{Re} \nabla^{*2} \vec{V}^{*} \tag{1.4.3}$$

$$\nabla^{*} \times \vec{B}^{*} = Rm(\vec{E}^{*} + \vec{V}^{*} \times \vec{B}^{*}) \tag{1.4.4}$$

$$\nabla^{*} \cdot \vec{B}^{*} = 0 \tag{1.4.5}$$

$$\nabla^{*} \times \vec{E}^{*} = -\frac{\partial \vec{B}^{*}}{\partial t^{*}} \tag{1.4.6}$$

with $\nabla^{*} \cdot \vec{E}^{*} = 0$ or $\nabla^{*} \cdot \vec{I}^{*} = 0$ (1.4.7)

where $\vec{I}^{*} = \vec{E}^{*} + \vec{V}^{*} \times \vec{B}^{*}$ (1.4.8)

The dimensionless flow and field parameters appearing in the above equations are defined as

$$Re = \frac{V_I(\Delta R)}{\nu} \cdot \left(\frac{\Delta V}{V_I}\right) \tag{1.4.9}$$

is the product of the hydrodynamic Reynolds number and a quantity closely related to the slip;

$$Rm = \sigma \mu_0 (\Delta R)(\Delta V) \qquad (1.4.10)$$

is the magnetic Reynolds number;

$$N = \frac{\sigma (\Delta R) B_0^{\,2}}{\rho (\Delta V)}$$

is the magnetic interaction parameter and it can be written as

$$N = S\, Rm \qquad (1.4.11)$$

where

$$S = \frac{B_0^{\,2}}{\mu_0 \rho (\Delta V)^2} \qquad (1.4.12)$$

is the magnetic pressure coefficient.

Equations (1.4.4) to (1.4.8) can be combined to give an alternate expression for the magnetic field:

$$\nabla^{*2} \vec{B}^{*} = Rm \left[\frac{\partial \vec{B}^{*}}{\partial t^{*}} - \nabla^{*} \times (\vec{V}^{*} \times \vec{B}^{*}) \right] \qquad (1.4.13)$$

For the sake of simplicity, we shall omit the asterisk from the dimensionless quantities throughout the following sections.

5. Series Expansion in Terms of Magnetic Reynolds Numbers

Since to obtain an exact solution for the basic set of equations (given by (1.4.2) to (1.4.8) would be most difficult for the given three-dimensional configuration, the application of an approximate method, such as series expansion in a characteristic field parameter, becomes unavoidable.

As it has been previously indicated, in MHD flows the magnitude of the magnetic Reynolds number indicates the interaction-intensity of the hydrodynamic and electromagnetic field distributions and usually $Rm \ll 1$ in MHD generator applications. The natural choice is, therefore, the series expansion of the various field variables in positive powers of the magnetic Reynolds number (Ref. [15]):

$$
\begin{aligned}
\vec{V} &= \vec{V}_o + Rm\,\vec{V}_1 + Rm^2\vec{V}_2 + \ldots \\
p &= p_o + Rm\,p_1 + Rm^2 p_2 + \ldots \\
\vec{B} &= \vec{B}_o + Rm\,\vec{B}_1 + Rm^2\vec{B}_2 + \ldots \qquad (1.5.1) \\
\vec{E} &= \vec{E}_o + Rm\,\vec{E}_1 + Rm^2\vec{E}_2 + \ldots \\
\vec{I} &= \vec{I}_o + Rm\,\vec{I}_1 + Rm^2\vec{I}_2 + \ldots
\end{aligned}
$$

These quantities can be substituted now in the basic set of equations and equating then the terms containing like powers of Rm, ordered sets of equations are obtained. Solution for each set of equations is obtained by utilizing the previous lower order solutions, as will be seen later.

The complete solutions for the various field distributions are obtained in the form of infinite series which are expected to converge for small values of Rm . The rate of convergence, however, is controlled by the magnitude of the magnetic pressure coefficient S , as can be seen from the momentum equation (1.4.3), where N = SRm .

For small magnetic Reynolds numbers, distinction should be made between the following cases: a) the magnetic pressure coefficient S is much larger than unity (strong magnetic fields coupled with moderate velocities); b) the magnetic pressure coefficient is of unit order or less.

For large values of S (case "a"), the product SRm in the momentum equation might be of unit order or larger ($N \geqslant O(1)$) even if the magnetic Reynolds number itself is small compared to unity; thus the omission of the electromagnetic term from the zeroth order equation could not be justified. In such cases, performing the series expansion in Rm the magnetic interaction parameter N should be retained in the momentum equation and treated as an independent parameter. As a result of such procedure, the zeroth order momentum equation will already contain an electromagnetic term.

If the magnetic pressure coefficient is of unit order (case "b"), then the order of the magnetic interaction parameter is defined by the magnitude of the magnetic Reynolds number: $N \sim O(Rm)$, and the electromagnetic term in the momentum equation becomes small compared to the other terms. If a series expansion in Rm is made and S is retained now as independent parameter, the zeroth order hydrodynamic equations will be completely uncoupled from the electromagnetic field equations. This is in full accord with the physical nature of the phenomenon discussed here: for $N \sim O(Rm) \ll 1$ the electromagnetic term in the momentum equation is small enough, so that its influence on the velocity distribution can be completely neglected in the first (= zeroth order) approximation.

The magnetic pressure coefficient, S , will be assumed to be of unit order throughout the present analysis so that full advantage of the mathematical simplification offered by the perturbation technique can be taken. The analysis will be restricted to the determination of the zeroth and first order field distributions, i.e., the first two terms in the series expansion. It should be mentioned here, however, that the same procedure could be applied to cases with moderately large values of S , but a greater number of terms of the series expansion would have to be calculated for satisfactory accuracy of the solution.

Substituting now (1.5.1) into the basic set of equations (1.4.2) to (1.4.8), and equating the terms containing like powers of Rm the following ordered sets of equations are obtained:

Zeroth order:

$$\nabla \cdot \vec{V}_o = 0 \qquad (1.5.2)$$

$$\frac{\partial \vec{V}_o}{\partial t} + (\vec{V}_o \cdot \nabla)\vec{V}_o = -\nabla p_o + Re^{-1}\nabla^2 \vec{V}_o \qquad (1.5.3)$$

$$\nabla \times \vec{E}_o = -\frac{\partial \vec{B}_o}{\partial t} \qquad (1.5.4)$$

with $\nabla \cdot \vec{E}_o = 0$ or $\nabla \cdot \vec{I}_o = 0$ (1.5.5)

where $\vec{I}_o = \vec{E}_o + \vec{V}_o \times \vec{B}_o$ (1.5.6)

The zeroth order magnetic induction field is completely determined by the applied magnetic field and it need not be considered here.

As can be seen, the hydrodynamic equations are completely uncoupled from the rest of the zeroth order set.

First order:

$$\nabla \cdot \vec{V}_1 = 0 \qquad (1.5.7)$$

$$\frac{\partial \vec{V}_1}{\partial t} + (\vec{V}_o \cdot \nabla)\vec{V}_1 + (\vec{V}_1 \cdot \nabla)\vec{V}_o = -\nabla p_1 + Re^{-1}\nabla^2\vec{V}_1 +$$

$$+ S\left[\vec{E}_o \times \vec{B}_o + (\vec{V}_o \times \vec{B}_o) \times \vec{B}_o\right] \qquad (1.5.8)$$

$$\nabla \times \vec{B}_1 = \vec{E}_o + \vec{V}_o \times \vec{B}_o \qquad (1.5.9)$$

$$\nabla \cdot \vec{B}_1 = 0 \qquad (1.5.10)$$

$$\nabla \times \vec{E}_1 = -\frac{\partial \vec{B}_1}{\partial t} \tag{1.5.11}$$

with $\nabla \cdot \vec{E}_1 = 0$ or $\nabla \cdot \vec{I}_1 = 0$ (1.5.12)

where $\vec{I}_1 = \vec{E}_1 + \vec{V}_o \times \vec{B}_1 + \vec{V}_1 \times \vec{B}_o$ (1.5.13)

The second order equations are as follows:

$$\nabla \cdot \vec{V}_2 = 0 \tag{1.5.14}$$

$$\frac{\partial \vec{V}_2}{\partial t} + (\vec{V}_o \cdot \nabla)\vec{V}_2 + (\vec{V}_1 \cdot \nabla)\vec{V}_1 + (\vec{V}_2 \cdot \nabla)\vec{V}_o = -\nabla p_2 +$$

$$+ Re^{-1}\nabla^2\vec{V}_2 + S\Big[\vec{E}_o \times \vec{B}_1 + \vec{E}_1 \times \vec{B}_o + (\vec{V}_o \times \vec{B}_o) \times \vec{B}_1 +$$

$$+ (\vec{V}_o \times \vec{B}_1) \times \vec{B}_o + (\vec{V}_1 \times \vec{B}_o) \times \vec{B}_o\Big] \tag{1.5.15}$$

$$\nabla \times \vec{B}_2 = \vec{E}_1 + \vec{V}_o \times \vec{B}_1 + \vec{V}_1 \times \vec{B}_o \tag{1.5.16}$$

$$\nabla \cdot \vec{B}_2 = 0 \tag{1.5.17}$$

$$\nabla \times \vec{E}_2 = -\frac{\partial \vec{B}_2}{\partial t} \tag{1.5.18}$$

with $\nabla \cdot \vec{E}_2 = 0$ or $\nabla \cdot \vec{I}_2 = 0$ (1.5.19)

where $\vec{I}_2 = \vec{E}_2 + \vec{V}_o \times \vec{B}_2 + \vec{V}_1 \times \vec{B}_1 + \vec{V}_2 \times \vec{B}_o$ (1.5.20)

Using the same method higher order sets can also be obtained.

6. The Choice of Coordinate System

There are two possibilities in choosing the coordinate system, neither of them offers any particular advantages relative to the other.

One of the possibilities is a space-fixed coordinate system. Since the boundaries of the generator device discussed here are axisymmetric, the obvious choice is the cylindrical-polar coordinate system fixed with respect to the inner (stationary) cylinder. The z-axis is directed along the axis of the concentric cylinders and the end-plates are given by the coordinates $z = 0$ and $z = L$. The applied magnetic field is described in this system as

$$B_o = \hat{r}\cos(\omega t - \phi) + \hat{\phi}\sin(\omega t - \phi) \qquad (1.6.1)$$

where $\omega/2\pi$ is the frequency of the "rotating" magnetic field. The field equations (with the exception of the zeroth order hydrodynamic equations) are time-dependent in this system as well as functions of the azimuthal coordinate "ϕ".

The second choice of the coordinate system is suggested by the nature of the electromagnetic induction phenomenon. As was pointed out previously, the various field quantities depend on the magnitude of the "slip" velocity rather than on the absolute magnitude of the velocities. Since the magnetic field appears to "rotate" with a uniform velocity, one can fix the coordinate system to this rotating field thus eliminating the time dependence from the basic equations. Unfortunately the azimuthal dependence still remains as can be seen from the transformation formulas given below. If the quantities measured in the rotating system are denoted by primes, then the following correlations hold (Fig. 3):

$$
\begin{aligned}
r' &= r \\
\phi' &= \phi - \omega t \\
z' &= z \\
V_r' &= V_r \\
V_\phi' &= V_\phi - \omega r \\
V_z' &= V_z
\end{aligned}
\qquad (1.6.2)
$$

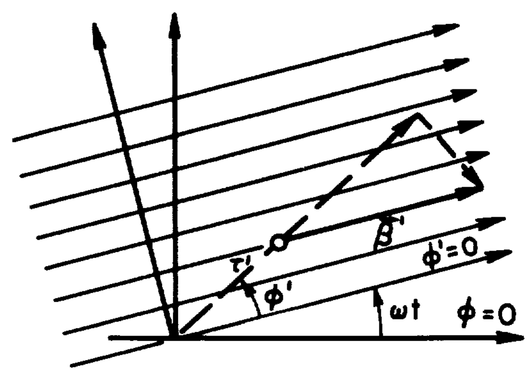


Figure 3

and the time independent magnetic field in the rotating system is described as

$$\vec{B}' = \hat{r}' \cos \phi' - \hat{\phi}' \sin \phi' \qquad (1.6.3)$$

The curl of the electric field vanishes in the rotating coordinate system (see Equation 1.4.6), thus the basic equations can be written in a somewhat simpler form.

When the inverse transformation is made, however, from the rotating to a space fixed coordinate system, an additional electric field must be calculated due to the relative motion of the two coordinate system (Ref. [16], [17]). Hence, neither of the coordinate systems offers particular advantages relative to the other in mathematical sense.

In different parts of the analysis, however, one coordinate system may be more convenient than the other and each will be used accordingly.

II. THE ZEROTH ORDER FIELD DISTRIBUTIONS

The solution of the zeroth order set of equations will be obtained in a space-fixed coordinate system.

1.) The Hydrodynamic Field

The zeroth order hydrodynamic equations corresponding to the conditions outlined in the introduction can be written in expanded form as

$$\frac{\partial}{\partial r}(r\, V_{ro}) + \frac{\partial V_{\phi o}}{\partial \phi} = 0 \tag{2.1.1}$$

$$\frac{\partial V_{ro}}{\partial t} + V_{ro}\frac{\partial V_{ro}}{\partial r} + \frac{V_{\phi o}}{r}\frac{\partial V_{ro}}{\partial \phi} - \frac{V_{\phi o}^{2}}{r} = -\frac{\partial p_o}{\partial r} + $$

$$+ \mathrm{Re}^{-1}\left(\frac{\partial^2 V_{ro}}{\partial r^2} + \frac{1}{r}\frac{\partial V_{ro}}{\partial r} - \frac{V_{ro}}{r^2} + \frac{1}{r^2}\frac{\partial^2 V_{ro}}{\partial \phi^2} - \frac{2}{r^2}\frac{\partial V_{\phi o}}{\partial \phi}\right) \tag{2.1.2}$$

$$\frac{\partial V_{\phi o}}{\partial t} + V_{ro}\frac{\partial V_{\phi o}}{\partial r} + \frac{V_{\phi o}}{r}\frac{\partial V_{\phi o}}{\partial \phi} + \frac{V_{ro}V_{\phi o}}{r} = -\frac{1}{r}\frac{\partial p_o}{\partial \phi} + $$

$$+ \mathrm{Re}^{-1}\left(\frac{\partial^2 V_{\phi o}}{\partial r^2} + \frac{1}{r}\frac{\partial V_{\phi o}}{\partial r} - \frac{V_{\phi o}}{r^2} + \frac{1}{r^2}\frac{\partial^2 V_{\phi o}}{\partial \phi^2} + \frac{2}{r^2}\frac{\partial V_{ro}}{\partial \phi}\right) \tag{2.1.3}$$

with the boundary conditions

$$V_{\phi o} = V_I \quad \text{at} \quad r = \frac{R_I}{\Delta R} \equiv \rho_I \tag{2.1.4}$$

$$p_o = p_I \tag{2.1.5}$$

$$V_{\phi o} = 0 \tag{2.1.6}$$

$$\text{at} \quad r = \frac{R_o}{\Delta R} \equiv \rho_o$$

$$V_{ro} = V_o \tag{2.1.7}$$

where V_I, V_o, p_I are dimensionless quantities; $V_o \equiv \frac{1}{\Delta V}(Q_o/A_o)$; Q_o being the total volume flow and A_o the total area of the exit ports.

Since the electromagnetic field does not affect the zeroth order velocity distribution and the boundary conditions imposed on the hydrodynamic field are steady, axisymmetric, the quantities described by (2.1.1) to (2.1.3) will depend neither on time nor the azimuthal coordinate.

Under such conditions Eq. (2.1.1) can be solved at once:

$$V_{ro} = \frac{\text{const.}}{r} \tag{2.1.8}$$

of applying the boundary condition (2.1.7)

$$V_{ro} = \frac{Q}{r} \quad , \text{ where } \quad Q \equiv -\rho_o V_o \tag{2.1.9}$$

Equation (2.1.3) can be written now:

$$\frac{d^2V_{\phi o}}{dr^2} + \frac{1}{r}\,\frac{dV_{\phi o}}{dr}\,(1 - QRe) - \frac{V_{\phi o}}{r^2}\,(1 + QRe) = 0 \tag{2.1.10}$$

The general solution of (2.1.10) is represented by:

$$V_{\phi o} = ar^D + br^{-1} \tag{2.1.11}$$

$$\text{where} \quad D \equiv 1 + QRe$$

The constants a and b can be determined by applying (2.1.4) and (2.1.6):

$$\left.\begin{aligned} a\rho_I^{\,D} + b\,\rho_I^{\,-1} &= V_I \\ a\rho_o^{\,D} + b\,\rho_o^{\,-1} &= 0 \end{aligned}\right\} \tag{2.1.12}$$

Hence

$$a = \frac{1}{\rho_I^{\,D}}\,\frac{V_I}{(1 - \beta^{D+1})}\ ; \quad b = -\,\rho_o\beta^D\,\frac{V_I}{(1 - \beta^{D+1})} \tag{2.1.13}$$

where

$$\beta \equiv \frac{\rho_o}{\rho_I} = \frac{R_o}{R_I} \tag{2.1.14}$$

Thus the viscous solution for the zeroth order azimuthal velocity can be written as

$$V_{\phi o} = \frac{V_I}{(1 - \beta^{D+1})} \left[\left(\frac{r}{\rho_I} \right)^D - \beta^D \left(\frac{\rho_o}{r} \right) \right] \qquad (2.1.15)$$

The expression for $V_{\phi o}$ has the familiar structure characterizing the cylindrical Couette-flows: the polynomial contains two terms, one being directly-, and the other inversely proportional to some power of r.

The zeroth order pressure distribution is determined by (2.1.2).

$$\frac{dp_o}{dr} = \frac{V_{\phi o}^{\ 2}}{r} - V_{ro} \frac{\partial V_{ro}}{\partial r} = \frac{V_{\phi o}^{\ 2}}{r} + \frac{Q^2}{r^3} =$$

$$= \frac{Q^2}{r^3} + \frac{V_I^{\ 2}}{(1 - \beta^{D+1})^2} \left[\frac{r^{2D-1}}{\rho_I^{\ 2D}} - 2 \left(\frac{\beta}{\rho_I} \right)^D \rho_o r^{D-2} + \beta^{2D} \frac{\rho_o^{\ 2}}{r^3} \right]$$

(2.1.16)

Hence

$$p_o = \text{const.} - \frac{Q^2}{2r^2} + \frac{V_I^{\ 2}}{(1 - \beta^{D+1})^2} \left[\frac{r^{2D}}{2D \rho_I^{\ 2D}} - \right.$$

$$\left. - \frac{2 \rho_o}{D - 1} \left(\frac{\beta}{\rho_I} \right)^D r^{D-1} - \frac{\rho_o^{\ 2} \beta^{2D}}{2r^2} \right] \qquad (2.1.17)$$

The complete solution can be obtained by applying the condition (2.1.5):

$$p_o = p_I - \frac{Q^2}{2}\left(\frac{1}{r^2} - \frac{1}{\rho_I^2}\right) - \frac{V_I^{\,2}}{(1-\beta^{D+1})^2}\cdot\left[\frac{\rho_I^{\,2D} - r^{2D}}{2D\cdot\rho_I^{\,2D}} - \right.$$

$$\left. - \frac{2\rho_o}{D-1}\left(\frac{\beta}{\rho_I}\right)^D(\rho_I^{\,D-1} - r^{D-1}) + \frac{\rho_o^{\,2}\beta^{2D}}{2}\left(\frac{1}{r^2} - \frac{1}{\rho_I^2}\right)\right] =$$

$$= p_I - \frac{Q^2}{2\rho_I^{\,2}}\cdot\left(\frac{1}{r_I^{\,2}} - 1\right) - \frac{V_I^{\,2}}{(1-\beta^{D+1})^2}\left[\frac{1}{2D}\,(1 - r_I^{\,2D}) + \right.$$

$$\left. + \frac{\beta^{2(D+1)}}{2}\left(\frac{1}{r_I^{\,2}} - 1\right) - \frac{2\beta^{D+1}}{D-1}\,(1 - r_I^{\,D-1})\right] \qquad (2.1.18)$$

where r_I is defined as

$$r_I \equiv r/\rho_I = \frac{r/\Delta R}{R_I/\Delta R} = \frac{r}{R_I} \qquad (2.1.19)$$

As will be seen later, the presence of the (r^D) term in the expression for the viscous velocity distribution (D is a constant proportional to the hydrodynamic Reynolds number) introduces substantial difficulties in the procedure of determination of the corresponding electromagnetic fields, especially in case of large Reynolds numbers.

Therefore, the inviscid solution will be used and this can be obtained from (2.1.1) to (2.1.3) by considering the limiting case when $Re \rightarrow \infty$. The above equations can be rewritten now as

$$V_{ro} \frac{dV_{\phi o}}{dr} + \frac{V_{ro} V_{\phi o}}{r} = 0 \qquad (2.1.20)$$

with

$$V_{\phi o} = V_I \quad \text{at} \quad r = \rho_I$$

(The no-slip boundary condition is dropped.)

$$\frac{dp_o}{dr} = \frac{V_{\phi o}^{\,2}}{r} - V_{ro} \frac{dV_{ro}}{dr} \qquad (2.1.21)$$

with

$$p_o = p_I \quad \text{at} \quad r = \rho_I$$

The solution of the zeroth order radial velocity remains unaltered.

Equations (2.1.20) and (2.1.21) are satisfied by the following solution:

$$V_{\phi o} = \frac{\rho_I V_I}{r} = \frac{V_I}{r_I} \qquad (2.1.22)$$

$$p_o = p_I - \frac{1}{2\rho_I^{\,2}} (V_I^{\,2} + Q^2)\left(\frac{1}{r_I^{\,2}} - 1\right) \qquad (2.1.23)$$

Equations (2.1.22) and (2.1.23) are the well-known potential vortex solutions.

2.) The Electric Field

The dimensionless equations (1.5.4), (1.5.5) describing the zeroth order electric field can be written in expanded form as follows:

$$\frac{1}{r}\frac{\partial E_{zo}}{\partial \phi} - \frac{\partial E_{\phi o}}{\partial z} = \omega \sin(\omega t - \phi)$$

$$\frac{\partial E_{ro}}{\partial z} - \frac{\partial E_{zo}}{\partial r} = -\omega \cos(\omega t - \phi) \qquad (2.2.1)$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rE_{\phi o}) - \frac{1}{r}\frac{\partial E_{ro}}{\partial \phi} = 0$$

$$\frac{\partial}{\partial r}(rE_{ro}) + \frac{1}{r}\frac{\partial E_{\phi o}}{\partial \phi} + \frac{\partial E_{zo}}{\partial z} = 0 \qquad (2.2.2)$$

We shall assume that the equation (2.2.2) is satisfied in the entire flow region except in a thin layer at the boundaries where a charge accumulation is assumed to take place. As a consequence of this hypothetical charge distribution, the normal component of the current vanishes at the non-conducting boundaries. The electrostatic field generated by this charge distribution is irrotational hence it can be described by a potential function.

Accordingly, it can be assumed that

$$\vec{E}_0 = \vec{E}_{01} + \vec{E}_{02} \qquad (2.2.3)$$

where $\vec{E}_{01}$ is governed by (2.2.1), $\vec{E}_{02}$ is the electrostatic field and is described as

$$\vec{E}_{02} = - \nabla \phi_0 \quad \text{with} \quad \nabla^2 \phi_0 = 0 \qquad (2.2.4)$$

The boundary conditions on ϕ_0 are defined by the restrictions imposed on the current distribution (1.5.6) at the nonconducting boundaries:

$$I_{ro} = E_{r01} - \frac{\partial \phi_0}{\partial r} + (\vec{V}_0 \times \vec{B}_0)_r = 0 \qquad (2.2.5)$$

$$\text{at } r = \rho_0 \text{ and } r = \rho_I$$

$$I_{zo} = E_{z01} - \frac{\partial \phi_0}{\partial z} + (\vec{V}_0 \times \vec{B}_0)_z = 0 \qquad (2.2.6)$$

$$\text{at } z = 0 \text{ and } z = L$$

Thus the boundary conditions on ϕ_0 in explicit form:

$$\frac{\partial \phi_0}{\partial r} = E_{r01} \quad \text{at} \quad r = \rho_0 \; ; \; r = \rho_I \qquad (2.2.5a)$$

$$\frac{\partial \phi_0}{\partial z} = E_{z01} + V_{ro} B_{\phi o} - V_{\phi o} B_{ro} \qquad (2.2.6a)$$

$$\text{at } z = 0 \; ; \; z = L$$

Furthermore, for the given configuration, ϕ_0 cannot be a discontinuous function of the azimuthal coordinate ϕ .

In the following, solutions will be obtained for the electric fields $\vec{E}_{01}$ and $\vec{E}_{02}$.

The set of equations describing $\vec{E}_{01}$ (2.2.1) is satisfied by a solution of the following form:

$$E_{r01} = E_{\phi 01} = 0 \;;\; E_{z01} = \omega r \cos(\omega t - \phi) \tag{2.2.7}$$

The two vanishing field components are explained by the well known physical principle that the induced electromotive force is perpendicular to the plane in which the magnetic flux lines "cut" the conductor.

Next, the field $\vec{E}_{02}$ given by the potential function $\bar{\Phi}_o$ will be determined.

The boundary conditions on $\bar{\Phi}_o$ are rewritten now as

$$\frac{\partial \bar{\Phi}_o}{\partial r} = 0 \text{ at } r = \rho_o \;;\; r = \rho_I \tag{2.2.8}$$

$$\frac{\partial \bar{\Phi}_o}{\partial z} = (\omega r - V_{\phi o}) \cos(\omega t - \phi) + V_{ro} \sin(\omega t - \phi) \tag{2.2.9}$$

The boundary conditions (2.2.8), (2.2.9) together with the basic equation $\nabla^2 \bar{\Phi}_o = 0$, define $\bar{\Phi}_o$ as an analytic function in the region $\rho_o \leq r \leq \rho_I$, $0 \leq z \leq L$ whose normal derivatives are specified on the surfaces bounding the region.

The boundary value problem thus defined is an example of the Neumann problem and its solution is straightforward if the corresponding modified Green's function (= Neumann function) is found for the given configuration.

The difficulties connected with the construction of the Neumann function for a finite cylinder suggest the use of some other method for the solution of the given problem. The application of some integral transformation appears to be most convenient, as will be seen later.

Since the boundary conditions contain both $\sin(\omega t - \phi)$ and $\cos(\omega t - \phi)$ terms, it is assumed that

$$\bar{\Phi}_0 = \bar{\Phi}_{01} \cos(\omega t - \phi) + \bar{\Phi}_{02} \sin(\omega t - \phi) \qquad (2.2.10)$$

where $\bar{\Phi}_{01}$ and $\bar{\Phi}_{02}$ must satisfy the equations

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2} + \frac{\partial^2}{\partial z^2} \right) \bar{\Phi}_{01} = 0$$

$$\frac{\partial \bar{\Phi}_{01}}{\partial r} = 0 \quad \text{at} \quad r = \rho_0 \ ; \ r = \rho_I \qquad (2.2.11)$$

$$\frac{\partial \bar{\Phi}_{01}}{\partial z} = (\omega r - V_{\phi_0}) \quad \text{at} \quad z = 0 \ ; \ z = L ,$$

$$\left.\begin{aligned}
&\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2} + \frac{\partial^2}{\partial z^2}\right)\bar{\phi}_{02} = 0 \\
&\frac{\partial \bar{\phi}_{02}}{\partial r} = 0 \quad \text{at} \quad r = \rho_0 \; ; \; r = \rho_I \\
&\frac{\partial \bar{\phi}_{02}}{\partial z} = V_{ro} \quad \text{at} \quad z = 0 \; ; \; z = L
\end{aligned}\right\} \quad (2.2.12)$$

Since the normal derivatives of the functions $\bar{\phi}_{01}$ and $\bar{\phi}_{02}$ are given at the endpoints of the z-interval, a finite Fourier cosine transform will be applied with respect to the z-coordinate.

The transformation coordinates are defined as

$$r_1 \equiv \frac{\pi}{L} r = mr \qquad m \equiv \frac{\pi}{L} \qquad (2.2.13)$$

$$z_1 \equiv \frac{\pi}{L} z = mz$$

The finite Fourier cosine transforms of $\bar{\phi}_{01}$ and $\bar{\phi}_{02}$ will be defined now as

$$C_{1n}\left[\bar{\phi}_{01}\right] \equiv \sqrt{\frac{2}{\pi}} \int_0^{\pi} \bar{\phi}_{01}(r_1, z_1) \cdot \cos(nz_1) dz_1 \quad (2.2.14)$$

$$C_{2n}\left[\bar{\phi}_{02}\right] = \sqrt{\frac{2}{\pi}} \int_0^{\pi} \bar{\phi}_{02}(r_1, z_1) \cdot \cos(nz_1) dz_1 \quad (2.2.15)$$

Then the functions Φ_{01} and Φ_{02} are given as

$$\Phi_{01} = \frac{C_{10}}{\sqrt{2\pi}} + \sqrt{\frac{2}{\pi}} \cdot \sum_{n=0}^{\infty} C_{1n} \cdot \cos(nz_1) \qquad (2.2.16)$$

$$\Phi_{02} = \frac{C_{20}}{\sqrt{2\pi}} + \sqrt{\frac{2}{\pi}} \cdot \sum_{n=0}^{\infty} C_{2n} \cdot \cos(nz_1) \qquad (2.2.17)$$

Note that

$$\frac{\partial C_{in}}{\partial r_1} = \sqrt{\frac{2}{\pi}} \cdot \int_0^{\pi} \frac{\partial \Phi_{oi}}{\partial r_1} \cos(nz_1) dz_1 = C_{in}\left[\frac{\partial \Phi_{oi}}{\partial r_1}\right] \qquad (2.2.18)$$

$$\frac{\partial^2 C_{in}}{\partial r_1^2} = \sqrt{\frac{2}{\pi}} \cdot \int_0^{\pi} \frac{\partial^2 \Phi_{oi}}{\partial r_1^2} \cos(nz_1) dz_1 = C_{in}\left[\frac{\partial^2 \Phi_{oi}}{\partial r_1^2}\right]$$

$$i = 1, 2 . \qquad (2.2.19)$$

Furthermore,

$$C_{in}\left[\frac{\partial^2 \Phi_{oi}}{\partial z_1^2}\right] = \sqrt{\frac{2}{\pi}} \int_0^{\pi} \frac{\partial^2 \Phi_{oi}}{\partial z_1^2} \cos(nz_1) dz_1 =$$

$$= (-1)^n \sqrt{\frac{2}{\pi}} \left. \frac{\partial \Phi_{oi}}{\partial z_1} \right|_{z_1=\pi} - \sqrt{\frac{2}{\pi}} \left. \frac{\partial \Phi_{oi}}{\partial z_1} \right|_{z_1=0} - n^2 C_{in}\left[\Phi_{oi}\right]$$

$$(2.2.20)$$

In the present case

$$\left.\frac{\partial \Phi_{oi}}{\partial z_1}\right|_{z_1=\pi} = \left.\frac{\partial \Phi_{oi}}{\partial z_1}\right|_{z_1=0} \quad \text{for} \quad i = 1,2 \quad .$$

Thus the equations (2.2.11) and (2.2.12) can be transformed into

$$\frac{d^2C_{1n}}{dr_1^{\,2}} + \frac{1}{r_1}\frac{dC_{1n}}{dr_1} - (n^2 + \frac{1}{r_1^{\,2}})C_{1n} = \left[1 - (-1)^n\right]\sqrt{\frac{2}{\pi}}\left(\frac{\omega}{m^2}r_1 - \frac{V_{\phi o}}{m}\right)$$

with $\dfrac{\partial C_{1n}}{\partial r_1} = 0$ at $r_1 = m\rho_o$; $r_1 = m\rho_I$ (2.2.21)

$$\frac{d^2C_{2n}}{dr_1^{\,2}} + \frac{1}{r_1}\frac{dC_{2n}}{dr_1} - (n^2 + \frac{1}{r_1^{\,2}})C_{2n} = \left[1 - (-1)^n\right]\sqrt{\frac{2}{\pi}}\left(-\frac{V_{ro}}{m}\right)$$

with $\dfrac{\partial C_{2n}}{\partial r_1} = 0$ at $r_1 = m\rho_o$; $r_1 = m\rho_I$ (2.2.22)

The summation index appears as parameter; therefore, three separate cases will be considered now.

a) For $n = 0 \quad (1 - (-1)^n) \equiv 0$

$$\left.\begin{aligned} C_{10} &= a_o r_1 + b_o r_1^{-1} \\ C_{20} &= c_o r_1 + d_o r_1^{-1} \end{aligned}\right| \qquad (2.2.23)$$

with
$$\left.\begin{aligned} a_o - b_o \cdot \rho_o^{-2} &= 0 \\ a_o - b_o \cdot \rho_I^{-2} &= 0 \\ c_o - d_o \cdot \rho_o^{-2} &= 0 \\ c_o - d_o \cdot \rho_I^{-2} &= 0 \end{aligned}\right| \qquad (2.2.24)$$

Hence $\quad C_{10} = C_{20} = 0 \qquad (2.2.25)$

b) For $n = 2, 4, 6, \ldots 2k$

$k = 1, 2, 3, 4, \ldots \qquad (2.2.26)$

$1 - (-1)^n = 0$ and the differential equations

(2.1.22) and (2.1.23) can be written as

$$\begin{aligned} \frac{d^2 C_{1n}}{dr_1^2} + \frac{1}{r_1}\frac{dC_{1n}}{dr_1} - \left(n^2 + \frac{1}{r_1^2}\right)C_{1n} &= 0 \\ \frac{d^2 C_{2n}}{dr_1^2} + \frac{1}{r_1}\frac{dC_{2n}}{dr_1} - \left(n^2 + \frac{1}{r_1^2}\right)C_{2n} &= 0 \end{aligned} \qquad (2.2.27)$$

$$\frac{dC_{1n}}{dr_1} = \frac{dC_{2n}}{dr_1} = 0 \quad \text{at} \quad r = m\,\rho_o \quad ; \quad r = m \cdot \rho_I$$

(2.2.27 Cont'd)

The differential equations with the corresponding boundary conditions given by (2.2.27) are examples of the general Sturm-Liouville problem (see Ref. [18] and [19]).

The differential equation for C_{1n} and C_{2n} together with the corresponding boundary conditions are all homogeneous, thus unique solutions to these equations cannot be obtained without specifying additional, nonhomogenous boundary conditions. Furthermore, since the boundary conditions specified for the electrostatic field and represented after the finite Fourier transforms by the RHS-s of equations (2.2.21) and (2.2.22) are not included any more in the equations (2.2.27), all even values of the index n will be omitted from further considerations.

c.) For $n = s = 2k + 1 = 1,3,5,7,\ldots$

$$k = 0,1,2,3,\ldots \qquad (2.2.28)$$

equations (2.2.22) and (2.2.23) can be rewritten as follows:

$$\frac{d^2C_{1s}}{dr_1^{\,2}} + \frac{1}{r_1}\frac{dC_{1s}}{dr_1} - \left(s^2 + \frac{1}{r_1^{\,2}}\right)C_{1s} = 2\sqrt{\frac{2}{\pi}}\cdot\left(\frac{\omega}{m^2}\,r_1 - \frac{V_{\phi o}}{m}\right)$$

$$\text{with}\ \frac{dC_{1s}}{dr_1} = 0 \quad \text{at} \quad r_1 = m\,\rho_o \ ; \ r_1 = m\,\rho_I \qquad (2.2.29)$$

$$\frac{d^2C_{2s}}{dr_1^{\,2}} + \frac{1}{r_1}\frac{dC_{2s}}{dr} - \left(s^2 + \frac{1}{r_1^{\,2}}\right)C_{2s} = 2\sqrt{\frac{2}{\pi}}\left(\frac{V_{ro}}{m}\right)$$

$$\text{with}\ \frac{dC_{2s}}{dr_1} = 0 \ \text{at}\ r_1 = m\,\rho_o \ ; \ r_1 = m\,\rho_I \qquad (2.2.30)$$

The complementary functions corresponding to the homogeneous LHS-s of equations (2.2.29) and (2.2.30) are:

$$(C_{1s})_c = A_s I_1\,(sr_1) + B_s\,K_1\,(sr_1) \qquad (2.2.31)$$

$$(C_{2s})_c = C_s I_1\,(sr_1) + D_s\,K_1\,(sr_1) \qquad (2.2.32)$$

The particular integrals corresponding to the RHS-s of the above equations can be computed for each case as

$$(C_{is})_p = \frac{1}{C}\int_{r_o}^{r}\left[R_1(t)R_2(r) - R_1(r)R_2(t)\right]t\,f_i(t)dt \qquad (2.2.33)$$

where

$i = 1, 2$

$$C = rW(R_1,R_2,r) = $$
$$= r\left[R_1(r)R_2'(r) - R_1'(r)R_2(r)\right]$$

W being the Wronskian of R_1 and R_2 ;

$R_1 \equiv I_1(sr)$

$R_2 \equiv K_1(sr)$

$f_i \equiv$ RHS of equations (2.2.29) or (2.2.30), respectively.

Hence the particular integral of eqn. (2.2.29) contains such terms as

$$\int_{r_o}^{r} t\, V_{\phi o}(t)\, I_1(st)dt \text{ and } \int_{r_o}^{r} t\, V_{\phi o}(t)\, K_1(st)dt \qquad (2.2.34)$$

Consequently if the viscous velocity distribution is to be used integrals such as $\int_{r_o}^{r} t^{D+1}\, I_1(st)dt$ and $\int_{r_o}^{r} t^{D+1} K_1(st)dt$ would have to be evaluated, where D is defined by (2.1.11) as

$D = 1 + Q\, R_e$.

The recurrence formula for cylindrical function ($Z_n(r)$) in general is given as

$$\int_{r_o}^{r} t^N\, Z_n(t)dt = -\,(N^2-n^2)\cdot \int_{r_o}^{r} t^{N-1} Z_n(t)dt\ +$$

$$+ \Big[r^{N+1} Z_{n+1}(r) - (N-n) r^n Z_n(r) \Big]_{r_o}^{r} \text{, where } N$$

is an arbitrary number. One may assume that similar expression could be derived also for the modified Bessel-functions..

Obviously the number of terms required for the evaluation of the above integral (see the recurrence formula) approaches the numerical value of the hydrodynamic Reynolds number in the given case. Thus for flows with high Reynolds numbers the integration process clearly becomes impractical.

Because of these difficulties the "inviscid" velocity distribution will be used in the determination of the electromagnetic fields. Such an approximation must yield satisfactory gross results as far as the overall power generation is concerned if the hydrodynamic Reynolds number characterizing the system is sufficiently high and the flow region with pronounced surface effects (such as the boundary layer) is comparatively small (as in case of large scale generators).

Taking advantage of the inviscid velocity distribution given by (2.1.9), (2.1.22) the particular integrals of eqns. (2.2.29) and (2.2.30) can be written as

$$(C_{1s})_p = -\frac{2}{s^2}\sqrt{\frac{2}{\pi}}\left(\frac{\omega}{m}r_1 - \frac{\rho_I V_I}{r_1}\right) \tag{2.2.35}$$

$$(C_{2s})_p = -\frac{2}{s^2}\sqrt{\frac{2}{\pi}}\cdot\frac{Q}{r_1} \tag{2.2.36}$$

Hence the complete solutions for C_{1s} and C_{2s} can be written as

$$C_{1s} = A_s I_1(sr_1) + B_s K_1(sr_1) - \frac{2}{s^2}\cdot\sqrt{\frac{2}{\pi}}\cdot\left(\frac{\omega}{m}r_1 - \frac{\rho_I V_I}{r_1}\right) \tag{2.2.37}$$

$$C_{2s} = C_s I_1(sr_1) + D_s K_1(sr_1) - \frac{2}{s^2}\cdot\sqrt{\frac{2}{\pi}}\cdot\frac{Q}{r_1} \tag{2.2.38}$$

The integration constants A_s, B_s, C_s and D_s are defined for each value of s by the following sets of equations:

$$\left.\begin{aligned}
&A_s\left[sI_o(sm\,\rho_o) - \frac{I_1(sm\,\rho_o)}{m\,\rho_o}\right] - B_s\left[sK_o(sm\,\rho_o) + \frac{K_1(sm\,\rho_o)}{m\,\rho_o}\right] = \\
&\qquad = \frac{2}{s^2m^2}\sqrt{\frac{2}{\pi}}\left(\omega + \frac{V_I}{\rho_o\beta}\right) \\
&A_s\left[sI_o(sm\,\rho_I) - \frac{I_1(sm\rho_I)}{m\,\rho_I}\right] - B_s\left[sK_o(sm\,\rho_I) + \frac{K_1(sm\rho_I)}{m\,\rho_I}\right] = \\
&\qquad = \frac{2}{s^2m^2}\sqrt{\frac{2}{\pi}}\left(\omega + \frac{V_I}{\rho_I}\right)
\end{aligned}\right\} \quad (2.2.39)$$

$$\left.\begin{aligned}
&C_s\left[sI_o(sm\,\rho_o) - \frac{I_1(sm\rho_o)}{m\,\rho_o}\right] - D_s\left[sK_o(sm\,\rho_o) + \frac{K_1(sm\rho_o)}{m\,\rho_o}\right] = \\
&\qquad = -\frac{2}{s^2m^2}\sqrt{\frac{2}{\pi}}\left(\frac{Q}{\rho_o^2}\right) - \frac{+2}{s^2m^2}\sqrt{\frac{2}{\pi}}\frac{V_o}{\rho_o} \\
&C_s\left[sI_o(sm\,\rho_I) - \frac{I_1(sm\rho_I)}{m\,\rho_I}\right] - D_s\left[sK_o(sm\,\rho_I) + \frac{K_1(sm\rho_I)}{m\,\rho_I}\right] = \\
&\qquad = +\frac{2}{s^2m^2}\sqrt{\frac{2}{\pi}}\frac{\beta V_o}{\rho_I}
\end{aligned}\right\} \quad (2.2.40)$$

Hence the potential function for the irrotational part of the zeroth order electric field can be written in the following form:

$$\bar{\Phi}_o = \sqrt{\frac{2}{\pi}} \sum_s^{\infty} \left\{ \left[A_s I_1(smr) + B_s K_1(smr) - \right.\right.$$

$$\left. - \frac{2}{s^2 m} \sqrt{\frac{2}{\pi}} \left(\omega r - \frac{\rho_I V_I}{r}\right) \right] \cos(\omega t - \phi) +$$

$$\left. + \left[C_s I_1(smr) + D_s K_1(smr) - \frac{2}{s^2 m} \sqrt{\frac{2}{\pi} \frac{Q}{r}} \right] \sin(\omega t - \phi) \right\} \cos(smz) \qquad (2.2.41)$$

where $s = 1, 3, 5, 7, \ldots = 2k + 1 \;:\; k = 0, 1, 2, \ldots$ and the constants A_s, B_s, C_s, and D_s are given by equations (2.2.39) and (2.2.40).

Thus the zeroth order electric field can be computed on the basis of (2.2.3):

$$\vec{E}_o = \hat{k} r\omega \cos(\omega t - \phi) - \nabla\bar{\Phi}_o \qquad (2.2.42)$$

One may notice at this point that the solution obtained for $\bar{\Phi}_o$ in form of an infinite series does not converge to the value of the function $\bar{\Phi}_o$ at the endpoints of the interval (at $z = 0$ and $z = L$). The exact solution requires

$$\frac{\partial\bar{\Phi}_o}{\partial z} = f(r, \phi) \quad \text{at} \quad z = 0; \; z = L$$

(see equ. 2.2.6a). The solution obtained by a finite Fourier cosine transform cannot satisfy this condition because $\sin 0 = \sin s\pi = 0$ for any integer value of "s". This results from the application of a Fourier cosine transform to a piecewise continuous function. (A similar situation occurs, for example, when one tries to expand the unit stepfunction into a Fourier sine series. The series will converge to the function everywhere except at the end-points of the interval.)

Expanding the first term on the RHS of (2.2.42) into an infinite Fourier sine series:

$$r\omega \cos(\omega t - \phi) = \frac{4}{\pi} \sum_s \frac{1}{s} (r\omega \cos(\omega t - \phi)) \sin(smz) \tag{2.2.43}$$

$s = 1, 3, 5, 7, \ldots. 2k+1 \; ; \; k = 0, 1, 2, \ldots.$

The components of the zeroth order electric field can be written as follows:

$$E_{ro} = \sqrt{\frac{2}{\pi}} \sum_s (sm) \cos(smz) \Big\{\Big[A_s\Big(-I_o(smr) + \frac{I_1(smr)}{smr}\Big) + B_s\Big(K_o(smr) + \frac{K_1(smr)}{smr}\Big) + \frac{2}{s^3m^2}\sqrt{\frac{2}{\pi}}\Big(\omega + \frac{\rho_I V_I}{r^2}\Big)\Big] \cos(\omega t - \phi) + \Big[C_s\Big(-I_o(smr) + \frac{I_1(smr)}{smr}\Big) + D_s\Big(K_o(smr) + \frac{K_1(smr)}{smr}\Big) - \frac{2}{s^3m^2}\sqrt{\frac{2}{\pi}}\frac{Q}{r^2}\Big]\sin(\omega t-\phi)\Big\}$$

$$E_{\phi o} = \sqrt{\frac{2}{\pi}} \sum_s \cos(smz) \Big\{\Big[\frac{C_sI_1(smr)}{r} + \frac{D_sK_1(smr)}{r} - \frac{2}{s^2m}\sqrt{\frac{2}{\pi}}\frac{Q}{r^2}\Big]\cos(\omega t-\phi) - \Big[A_s\frac{I_1(smr)}{r} + B_s\frac{K_1(smr)}{r} - \frac{2}{s^2m}\sqrt{\frac{2}{\pi}}\Big(\omega - \frac{\rho_I V_I}{r^2}\Big)\Big]\sin(\omega t - \phi)\Big\} \tag{2.2.44.}$$

$$E_{zo} = \sqrt{\frac{2}{\pi}} \sum_s (sm) \sin (smz) \Big\{ \Big[A_s I_1(smr) + B_s K_1(smr) + $$

$$+ \frac{2}{s^2 m} \sqrt{\frac{2}{\pi}} \left(\frac{\rho_I V_I}{r}\right) \Big] \cos (\omega t - \phi) + \Big[C_s I_1(smr) + D_s K_1(smr) - $$

$$- \frac{2}{s^2 m} \sqrt{\frac{2}{\pi}} \frac{Q}{r} \Big] \sin (\omega t - \phi) \Big\} \qquad (2.2.44)$$

III. THE FIRST ORDER FIELD DISTRIBUTIONS

The first order or perturbation fields are defined by equations (1.5.7) to (1.5.13).

The mathematical analysis connected with the solution of these equations becomes quite involved due to the three-dimensional zeroth order electromagnetic fields and current distribution defining some of the first order quantities.

Since an exact solution to the system of equations described above does not appear to be available, approximate methods of solution will be used.

In order to obtain a significant reduction in the complexity of the problem, the end plate effects will be completely neglected in the first order solution. As was shown by I. Bernstein [12] such an approximation is well justified if the duct configuration is such that currents are deflected from their principal direction (the z-direction here) after a considerable flow length only. This condition is satisfied in the given case if the height of the generator chamber is large compared to the width of the annulus.

For sufficiently large (L/R_I) ratios the eddy currents (the r and ϕ current components in the given case) become negligible compared to the current flow in the z-direction. Consequently the complete neglect of end plate effects leads to a one-dimensional current distribution thus reducing the difficulties connected with the mathematical treatment of the problem.

1.) The First Order Velocity Field

The differential equations describing the first order velocity and pressure distributions can be written in the following form:

$$\frac{1}{r}\frac{\partial}{\partial r}(r\, v_{r1}) + \frac{1}{r}\frac{\partial v_{\phi 1}}{\partial \phi} = 0 \qquad (3.1.1)$$

$$\frac{\partial v_{r1}}{\partial t} + v_{ro}\frac{\partial v_{r1}}{\partial r} + v_{\phi o}\frac{\partial v_{r1}}{r\partial \phi} + v_{r1}\frac{\partial v_{ro}}{\partial r} + \frac{v_{\phi 1}}{r}\frac{\partial v_{ro}}{\partial \phi} -$$

$$-2\frac{v_{\phi o} v_{\phi 1}}{r} = -\frac{\partial p_1}{\partial r} + S\left[- E_{zo} B_{\phi o} - v_{ro} B_{\phi o}^{2} + v_{\phi o} B_{ro} B_{\phi o}\right] +$$

$$+ Re^{-1}\left(\frac{\partial^2 v_{r1}}{\partial r^2} + \frac{1}{r}\frac{\partial v_{r1}}{\partial r} - \frac{v_{r1}}{r^2} + \frac{1}{r^2}\frac{\partial^2 v_{r1}}{\partial \phi^2} - \frac{2}{r^2}\frac{\partial v_{\phi 1}}{\partial \phi}\right) \qquad (3.1.2)$$

$$\frac{\partial v_{\phi 1}}{\partial t} + v_{ro}\frac{\partial v_{\phi 1}}{\partial r} + \frac{v_{\phi o}}{r}\frac{\partial v_{\phi 1}}{\partial \phi} + v_{r1}\frac{\partial v_{\phi o}}{\partial r} + \frac{v_{\phi 1}}{r}\frac{\partial v_{\phi o}}{\partial \phi} + \frac{1}{r}(v_{ro}\, v_{\phi 1} +$$

$$+ v_{r1}\, v_{\phi o}) = -\frac{\partial p_1}{r\partial \phi} + S(E_{zo} B_{ro} - v_{\phi o} B_{ro}^{2} - v_{ro} B_{ro} B_{\phi o}) +$$

$$+ Re^{-1}\left(\frac{\partial^2 v_{\phi 1}}{\partial r^2} + \frac{1}{r}\frac{\partial v_{\phi 1}}{\partial r} - \frac{v_{\phi 1}}{r^2} + \frac{1}{r^2}\frac{\partial^2 v_{\phi 1}}{\partial \phi^2} + \frac{2}{r^2}\frac{\partial v_{r1}}{\partial \phi}\right) \qquad (3.1.3)$$

The momentum equation in the z-direction is completely omitted in view of the assumption described above.

Since the zeroth order solution satisfies the exact boundary conditions, the boundary conditions ascribed to the perturbation field quantities must be homogeneous:

$$v_{r1} = 0 \quad \text{at} \quad r = \rho_o\,, \tag{3.1.4}$$

$$v_{\phi 1} = 0 \quad \text{at} \quad r = \rho_o\,,\ r = \rho_I\,, \tag{3.1.5}$$

and for the pressure distribution

$$p_1 = 0 \quad \text{at} \quad r = \rho_I\,. \tag{3.1.6}$$

The electromagnetic terms, appearing on the RHS-s of equ.-s (3.1.2) and (3.1.3) are periodic functions of the nondimensional time t and the ϕ-coordinate. Therefore one cannot assume a priori an axisymmetric, time-independent flow pattern as was done in the zeroth order solution.

The presence of time and ϕ derivatives in equations (3.1.1) to (3.1.3) makes the attempt to obtain an exact solution for the hydrodynamic field distributions difficult. Thus the search for an approximate solution appears to be justified in the given case.

One way to obtain an approximate solution for the above equations is to solve them for the time average of the hydrodynamic quantities (velocity, pressure) taken over the period of the magnetic field-rotation. This procedure, however, eliminates the periodic dependence of the physical quantities on the ϕ-coordinate. Since useful power is produced only by those current components which are in phase with the voltage (a phase shift between the two field vectors $\vec{I}$ and $\vec{E}$ leads to the production of reactive power) the phase relations among the various hydrodynamic and electromagnetic field quantities have an important role in the energy considerations. Therefore the elimination of the periodic dependence by computing the time average of the various quantities would be impractical and misleading.

Another possible approach is to neglect the viscosity effects and to transform the governing equations into a coordinate system fixed relative to and rotating with the magnetic field. In this way the physical quantities measured in the rotating coordinate system will not be time dependent. The azimuthal dependence, however, will not be eliminated by this transformation.

Accepting the second approach for obtaining an approximate solution for the first order velocity-distribution the following transformation coordinates are introduced:

$$\left.\begin{aligned} r' &= r \qquad ; \qquad v_r' = v_r \\ \phi' &= \phi - \omega t \quad ; \quad v_\phi' = v_\phi - \omega r \end{aligned}\right\} \quad (3.1.7)$$

The primed system is rotating with a constant angular velocity ω.

The direction of the x'-axis coincides with the direction of the imposed zeroth order magnetic field and rotates together with it.

The zeroth order nondimensional field quantities transposed into the rotating coordinate system can be expressed as follows:

$$v_{ro}' = \frac{Q}{r'} \; ; \qquad B_{ro}' = \cos\phi'$$

$$v_{\phi o}' = \frac{\rho_I V_I}{r'} - \omega r' \; ; \qquad B_{\phi o}' = -\sin\phi'$$

$$E_{ro}' = E_{\phi o}' = E_{zo}' \equiv 0$$

$$I_{ro}' = I_{\phi o}' \equiv 0$$

$$I_{zo}' = v_{ro}' B_{\phi o}' - v_{\phi o}' B_{ro}' = -\frac{Q}{r'} \sin\phi' - \left(\frac{\rho_I V_I}{r'} - \omega r'\right)\cos\phi' \tag{3.1.8}$$

As one may notice, the zeroth-order electric field vanishes completely. This follows from the fact that the zeroth-order magnetic field is time-independent now and the absence of end-plates eliminates the build-up of an electrostatic field. (See the discussion following equ. (2.2.2)).

Substituting (3.1.8) into the basic equations and neglecting the viscous terms the following system of differential equations is obtained:

$$\frac{\partial}{\partial r'}(r' v_{r1}') + \frac{\partial v_{\phi 1}'}{\partial \phi'} = 0 \tag{3.1.9}$$

$$\frac{Q}{r'}\ \frac{\partial v_{r1}'}{\partial r'} - \left(\omega - \frac{\rho_I V_I}{r'^2}\right)\frac{\partial v_{r1}'}{\partial \phi'} - \frac{Q}{r'^2}\, v_{r1}' + 2\left(\omega - \frac{\rho_I V_I}{r'^2}\right)v_{\phi 1}' =$$

$$= -\frac{\partial p_1'}{\partial r'} + S\left[\left(\omega r' - \frac{\rho_I V_I}{r'}\right)\sin\phi' \cos\phi' - \frac{Q}{r'}\sin^2\phi'\right] \tag{3.1.10}$$

$$\frac{Q}{r'}\frac{\partial v_{\phi}'}{\partial r'} - \left(\omega - \frac{\rho_I V_I}{r'^2}\right)\frac{\partial v_{\phi 1}'}{\partial \phi'} - \left(\omega + \frac{\rho_I V_I}{r'^2}\right)v_{r1}' + \frac{1}{r'}\left(\frac{Q}{r'}\, v_{\phi 1}' -\right.$$

$$\left. - \omega r'\, v_{r1}' + \frac{\rho_I V_I}{r'}\, v_{r1}'\right) = -\frac{\partial p_1'}{r'\partial \phi'} + S\left[\left(\omega r' -\right.\right.$$

$$\left.\left. - \frac{\rho_I V_I}{r'}\right)\cos^2\phi' - \frac{Q}{r'}\sin\phi'\cos\phi'\right] \tag{3.1.11}$$

Next, the pressure term will be eliminated from the two equations above by cross differentiation.

For sake of simplicity the primes will be omitted from the different variables during the following, but it should be understood that all quantities are expressed in the rotating coordinate system.

Performing the differentiation and subtracting (3.1.10) from (3.1.11):

$$Q\left(\frac{\partial^2 v_{\phi 1}}{\partial r^2} + \frac{1}{r}\frac{\partial v_{\phi 1}}{\partial r} - \frac{v_{\phi 1}}{r^2} - \frac{1}{r}\frac{\partial^2 v_{r1}}{\partial r\partial \phi} + \frac{1}{r^2}\frac{\partial v_{r1}}{\partial \phi}\right) +$$

$$+ \omega\left(-\frac{\partial v_{\phi 1}}{\partial \phi} - r\frac{\partial^2 v_{\phi 1}}{\partial r\partial \phi} - 2v_{r1} - 2r\frac{\partial v_{r1}}{\partial r} + \frac{\partial^2 v_{r1}}{\partial \phi^2} - 2\frac{\partial v_{\phi 1}}{\partial \phi}\right) +$$

$$+ \rho_I V_I\left(-\frac{1}{r^2}\frac{\partial v_{\phi 1}}{\partial \phi} + \frac{1}{r}\frac{\partial^2 v_{\phi 1}}{\partial r\partial \phi} - \frac{1}{r^2}\frac{\partial^2 v_{r1}}{\partial \phi^2} + \frac{2}{r^2}\frac{\partial v_{\phi 1}}{\partial \phi}\right) =$$

$$= S(2\omega r \cos^2 \phi - \omega r \cos^2\phi + \omega r \sin^2\phi +$$

$$+ \frac{\rho_I V_I}{r} \cos 2 \phi + \frac{Q}{r} \sin 2 \phi) \qquad (3.1.12)$$

Performing the possible algebraic simplification and introducing a stream function ψ such that

$$v_{r1} = \frac{\partial\psi}{r\partial\phi} \quad ; \quad v_{\phi 1} = - \frac{\partial\psi}{\partial r} \qquad (3.1.13)$$

equation (3.1.12) can be rewritten as

$$Q\Big[- \frac{\partial^3\psi}{\partial r^3} - \frac{1}{r}\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r^2}\frac{\partial\psi}{\partial r} - \frac{1}{r}\frac{\partial}{\partial\phi}\Big(- \frac{1}{r^2}\frac{\partial\psi}{\partial\phi} + \frac{1}{r}\frac{\partial^2\psi}{\partial r\partial\phi} -$$

$$- \frac{1}{r^2}\frac{\partial\psi}{\partial\phi}\Big)\Big] + \omega\Big[\frac{\partial^2\psi}{\partial r\partial\phi} + r\frac{\partial}{\partial\phi}\Big(\frac{\partial^2\psi}{\partial r^2}\Big) - \frac{2}{r}\frac{\partial\psi}{\partial\phi} -$$

$$- 2r\Big(- \frac{1}{r^2}\frac{\partial\psi}{\partial\phi} + \frac{1}{r}\frac{\partial^2\psi}{\partial r\partial\phi}\Big) + \frac{1}{r}\frac{\partial^3\psi}{\partial\phi^3} + 2\frac{\partial^2\psi}{\partial r\partial\phi}\Big] +$$

$$+ \rho_I V_I\Big[- \frac{1}{r^2}\frac{\partial^2\psi}{\partial r\partial\phi} - \frac{1}{r}\frac{\partial}{\partial\phi}\Big(\frac{\partial^2\psi}{\partial r^2}\Big) - \frac{1}{r^3}\frac{\partial^3\psi}{\partial\phi^3}\Big] =$$

$$= - Q\Big(\frac{\partial^3\psi}{\partial r^3} + \frac{1}{r}\frac{\partial^2\psi}{\partial r^2} - \frac{1}{r^2}\frac{\partial\psi}{\partial r} - \frac{2}{r^3}\frac{\partial^2\psi}{\partial\phi^2} + \frac{1}{r^2}\frac{\partial^3\psi}{\partial r\partial\phi^2}\Big) -$$

$$- \Big(\frac{\rho_I V_I}{r} - \omega r\Big)\Big(\frac{\partial}{\partial\phi}\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial\phi}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^3\psi}{\partial\phi^3}\Big) =$$

$$= S\Big[\frac{Q}{r}\sin 2\phi + \Big(\frac{\rho_I V_I}{r} - \omega r\Big)\cos 2\phi + 2\omega r\cos^2\phi\Big]$$

$$(3.1.14)$$

With the consideration of the following identities

$$\frac{\partial}{\partial r}(\nabla^2\psi) = \frac{\partial}{\partial r}\left(\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\phi^2}\right) =$$

$$= \frac{\partial^3\psi}{\partial r^3} + \frac{1}{r}\frac{\partial^2\psi}{\partial r^2} - \frac{1}{r^2}\frac{\partial\psi}{\partial r} - \frac{2}{r^3}\frac{\partial^2\psi}{\partial\phi^2} + \frac{1}{r^2}\frac{\partial^3\psi}{\partial r\partial\phi^2} \tag{3.1.15}$$

$$\frac{\partial}{\partial\phi}(\nabla^2\psi) = \frac{\partial}{\partial\phi}\left(\frac{\partial^2\psi}{\partial r^2}\right) + \frac{1}{r}\frac{\partial}{\partial\phi}\left(\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^3\psi}{\partial\phi^3} \tag{3.1.16}$$

equation (3.1.14) can be rewritten as

$$Q\frac{\partial}{\partial r}\left(\frac{\nabla^2\psi}{S}\right) + \left(\frac{\rho_I V_I}{r} - \omega r\right)\frac{\partial}{\partial\phi}\left(\frac{\nabla^2\psi}{S}\right) =$$

$$= -\frac{Q}{r}\sin 2\phi - \frac{\rho_I V_I}{r}\cos 2\phi - \omega r \; ;$$

Introducing the notation

$$F \equiv \frac{\nabla^2\psi}{S} \quad ; \quad \frac{\rho_I V_I}{Q} \equiv M \quad ; \quad \frac{\omega}{\rho_I V_I} = \delta \tag{3.1.17}$$

The above equation can be rewritten in the following form:

$$\frac{\partial F}{\partial r} + M(r^{-1} - \delta r)\frac{\partial F}{\partial\phi} = -\frac{\sin 2\phi}{r} - \frac{M\cos 2\phi}{r} - \delta Mr \tag{3.1.18}$$

The structure of equation (3.1.18) suggests to seek a solution for F of the following form:

$$F = F_1(r)\sin 2\phi + F_2(r)\cos 2\phi + F_3(r) \qquad (3.1.19)$$

Hence the first order stream function, defined by equation (3.1.17) can be written as

$$\psi = S[\psi_1(r)\sin 2\phi + \psi_2(r)\cos 2\phi + \psi_3(r)] \qquad (3.1.20)$$

where ψ_1, ψ_2, and ψ_3 satisfy the following set of equations:

$$\begin{aligned} \psi_1'' + \frac{1}{r}\psi_1' - \frac{4}{r^2}\psi_1 &= F_1 \\ \psi_2'' + \frac{1}{r}\psi_2' - \frac{4}{r^2}\psi_2 &= F_2 \\ \psi_3'' + \frac{1}{r}\psi_3' &= F_3 \end{aligned} \qquad (3.1.21)$$

The velocity components are given then in the rotating coordinate system as

$$\begin{aligned} v_{r1} &= \frac{\partial\psi}{r\partial\phi} = 2S\left[\frac{\psi_1}{r}\cos 2\phi - \frac{\psi_2}{r}\sin 2\phi\right] \\ v_{\phi 1} &= -\frac{\partial\psi}{\partial r} = -S\left[\psi_1'\sin 2\phi + \psi_2'\cos 2\phi + \psi_3'\right] \end{aligned} \qquad (3.1.22)$$

The new velocity components can now be expressed in a space-fixed coordinate system as

$$\begin{aligned} v_{r1} &= 2S\left[\frac{\psi_2}{r}\sin 2(\omega t - \phi) + \frac{\psi_1}{r}\cos 2(\omega t - \phi)\right] \\ v_{\phi 1} &= S\left[\psi_1'\sin 2(\omega t - \phi) - \psi_2'\cos 2(\omega t - \phi)\right] - \psi_3' + \omega r; \end{aligned} \qquad (3.1.23)$$

The boundary conditions (equations 3.1.4 and 3.1.5) imply

$$\left.\begin{aligned} \psi_1 &= 0 \\ \psi_2 &= 0 \\ \psi_1' &= 0 \\ \psi_2' &= 0 \end{aligned}\right\} \text{ at } \quad r = \rho_o,$$

$$\left.\begin{aligned} \psi_1' &= 0 \\ \psi_2' &= 0 \end{aligned}\right\} \text{ at } \quad r = \rho_I \qquad (3.1.24)$$

$$\psi_3' = \frac{\omega}{S}\rho_o \quad \text{at} \quad r = \rho_o$$

$$\psi_3' = \frac{\omega}{S}\rho_I \quad \text{at} \quad r = \rho_I$$

Returning to equation (3.1.19) now, the functions F_1, F_2 and F_3 must satisfy the following conditions:

$$F_1' - 2M(r^{-1} - \delta r)\, F_2 = -\frac{1}{r} \qquad (3.1.25)$$

$$F_2' + 2M(r^{-1} - \delta r)\, F_1 = -\frac{M}{r} \qquad (3.1.26)$$

$$F_3' = -\delta M r \qquad (3.1.27)$$

where the primes indicate differentiation with respect to r. Equation (3.1.27) can be integrated at once:

$$F_3 = -\frac{\delta M}{2} r^2 + A_1 \qquad (3.1.28)$$

Introducing the notation

$$g(r) \equiv 2M(r^{-1} - \delta r) \qquad (3.1.29)$$

equations (3.1.25) and (3.1.26) yield the following relations:

$$F_2 = \frac{1}{g}\left(F_1' + \frac{1}{r}\right) \tag{3.1.30}$$

with
$$F_2' = \frac{d}{dr}\left(\frac{F_1'}{g}\right) + \frac{d}{dr}\left(\frac{1}{rg}\right)$$

and
$$\frac{d}{dr}\left(\frac{F_1'}{g}\right) + \frac{d}{dr}\left(\frac{1}{rg}\right) + gF_1 = -\frac{M}{r} \tag{3.1.31}$$

Equation (3.1.31) is a linear second order differential equation defining the function F_1 up to two integration constants and it can be written in an expanded form as

$$\begin{aligned}(r^{-1} - \delta r)\, F_1'' + (r^{-2} + \delta)\, F_1' + (r^{-1} - \delta r)^3\, F_1 &= \\ = -\frac{M}{r}(r^{-1} - \delta r)^2 - \frac{1}{r}(r^{-2} + \delta) + \frac{1}{r^2}(r^{-1} - \delta r)\end{aligned} \tag{3.1.32}$$

The function F_1, defined by equation (3.1.32), can be found by applying the series solution method known as the method of Frobenius to equation (3.1.32). (In fact, a complete solution for the first order velocity distribution has been found by applying the series expansion method. For brevity, however, the details of the analysis will not be given here).

Another way of obtaining a solution for F_1 is to apply the method of successive approximations (Picard's method) to equation (3.1.31). (See references [20], [21]).

Thus defining F_1 as

$$F_1(r) = \lim_{n \to \infty} f_n(r) \tag{3.1.33}$$

the sequence of function $f_n(r)$ is given by (see equation (3.1.31)):

$$\frac{d}{dr}\left(\frac{f_n'}{g}\right) = -\frac{M}{r} - \frac{d}{dr}\left(\frac{1}{rg}\right) - g\, f_{n-1} \tag{3.1.34}$$

With the choice of $f_o = 0$, the indicated integrations are performed, and the first two functions of the sequence f_n are found to be

$$\begin{aligned} f_1 = {} & M^2 \ln r\, (\delta r^2 - \ln r) + (C_{11} - 1)\ln r \\ & - \frac{\delta}{2}(C_{11} + M^2) r^2 + C_{12} \end{aligned} \tag{3.1.35}$$

$$\begin{aligned} f_2 = {} & M^2 \ln r\,(\delta r^2 - \ln r) - \frac{\delta}{2} M^2 r^2 + C_{12}(\delta r^2 \ln r - \\ & - (\ln r)^2 - \frac{\delta^2}{4} r^4) + \frac{\delta}{2} M^2 (C_{11} + M^2)(r^2 - \frac{3}{4}\delta r^4 + \\ & + \frac{1}{6}\delta^2 r^6) - M^2 (C_{11} - 1)(\frac{2}{3}(\ln r)^3 - \delta r^2 (\ln r)^2 + \\ & + \frac{1}{2}\delta^2 r^4 \ln r - \frac{3}{8}\delta^2 r^4) + M^4(\frac{1}{3}(\ln r)^4 - \frac{2}{3}\delta r^2 (\ln r)^3 + \\ & + \frac{1}{2}\delta^2 r^4 (\ln r)^2 - \frac{1}{6}\delta^3 r^6 \ln r - \frac{\delta^2}{16} r^4 + \frac{5}{72}\delta^3 r^6) - \ln r + \\ & + C_{21}(\ln r - \frac{\delta}{2} r^2) + C_{22}\ ; \end{aligned} \tag{3.1.36}$$

C_{11}, C_{12} and C_{21}, C_{22} are the integration constants corresponding to the first and second approximations, respectively.

The integration constants for the n-th approximation are found by determining the corresponding stream functions ψ_{1n}, ψ_{2n} and applying the boundary conditions (3.1.24).

The stream function corresponding to the second approximation can be written as

$$\psi = S[\psi_1 \sin 2\phi + \psi_2 \cos 2\phi + \psi_3] \quad ,$$

where

$$\begin{aligned}
\psi_1 = \; & A_1 r^2 + B_1 r^{-2} + C_{21} \frac{r^2}{8} [(\ell nr)^2 - \tfrac{1}{2} \ell nr - \tfrac{\delta}{3} r^2] + \\
& + C_{22} \frac{r^2}{4} \ell nr - \frac{r^2}{8} \ell nr (\ell nr - \tfrac{1}{2}) + \\
& + \frac{1}{12} (M^2 + C_{12}) r^2 [\delta r^2 \ell nr - (\ell nr)^3 + \tfrac{3}{4} (\ell nr)^2 - \tfrac{3}{8} \ell nr] \\
& - \frac{\delta}{72} (7M^2 + 4C_{12} + \frac{9\delta}{16} C_{12} r^2) r^4 + \frac{\delta M^2}{24} (C_{11} + M^2) \times \\
& \quad r^4 (1 - \frac{9\delta}{32} + \frac{\delta^2}{30} r^4) - \frac{M^2}{32} (C_{11} - 1) [\tfrac{4}{3} r^2 \{(\ell nr)^4 - \\
& - (\ell nr)^3 + \tfrac{3}{4} (\ell nr)^2 - \tfrac{3}{8} \ell nr\} - \frac{8\delta}{3} r^4 \{(\ell nr)^2 - \\
& - \tfrac{4}{3} \ell nr + \tfrac{13}{18}\} + \frac{\delta^2}{2} r^6 \ell nr - \frac{9\delta^2}{16} r^6] + \\
& + \frac{M^4}{32} [\tfrac{2}{3} r^2 \{ \tfrac{4}{5} (\ell nr)^5 - (\ell nr)^4 + (\ell nr)^3 + \tfrac{3}{4} (\ell nr)^2 + \\
& + \tfrac{3}{8} \ell nr\} - \frac{16\delta}{9} r^4 \{(\ell nr)^3 - 2(\ell nr)^2 + \tfrac{13}{6} \ell nr - \tfrac{10}{9}\} \\
& + \frac{\delta^2}{2} r^6 \ell nr (\ell nr - \tfrac{3}{4}) - \frac{4\delta^3}{45} r^8 \ell nr + \frac{3\delta^2}{64} r^6 + \frac{41\delta^3}{15 \cdot 45} r^8]
\end{aligned} \tag{3.1.37}$$

$$\psi_2 = A_2r^2 + B_2r^{-2} + C_{21}\frac{r^2}{A}\ln r - C_{12}\frac{r^2}{8}[(\ln r)^2 -$$

$$- \frac{1}{2}\ln r - \frac{\delta}{3}r^2] - M\frac{r^2}{8}\ln r(\ln r - \frac{1}{2}) + \frac{\delta M}{8}(C_{11} + M^2) \times$$

$$\times (\frac{1}{3} - \frac{\delta r^2}{16})r^4 - \frac{M}{4}(C_{11} - 1)r^2[\frac{1}{3}(\ln r)^3 - \frac{1}{4}(\ln r)^2 +$$

$$+ \frac{1}{8}\ln r - \frac{\delta}{3}r^2\ln r + \frac{7\delta}{18}r^2] + \frac{M^3}{24}r^2[(\ln r)^4 - (\ln r)^3 +$$

$$+ \frac{3}{4}(\ln r)^2 - \frac{3}{8}\ln r - 2\delta r^2\ln r(\ln r - \frac{4}{3}) + \frac{3}{8}\delta^2r^4\ln r -$$

$$- \delta r^2(\frac{13}{9} + \frac{15}{64}\delta r^2)] \qquad (3.1.38)$$

$$\psi_3 = A_3\ln r + \frac{B_3}{4}r^2 - \frac{\delta M}{32}r^4 \qquad (3.1.39)$$

The integration constants of the second approximation A_1, A_2, A_3, B_1, B_2, B_3, C_{21} and C_{22} are found by solving the eight simultaneous algebraic equations (3.1.24), as was pointed out previously.

2.) The First Order Magnetic Field

The set of equations describing the perturbation magnetic field is:

$$\frac{1}{r}\frac{\partial B_{z1}}{\partial \phi} - \frac{\partial B_{\phi 1}}{\partial z} = E_{ro} = -\frac{\partial \phi_o}{\partial r} \tag{3.2.1}$$

$$\frac{\partial B_{r1}}{\partial z} - \frac{\partial B_{z1}}{\partial r} = E_{\phi o} = -\frac{\partial \phi_o}{r\partial \phi} \tag{3.2.2}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rB_{\phi 1}) - \frac{1}{r}\frac{\partial B_{r1}}{\partial \phi} = E_{zo} + v_{ro}B_{\phi o} - v_{\phi o}B_{ro} \tag{3.2.3}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rB_{r1}) + \frac{1}{r}\frac{\partial B_{\phi 1}}{\partial \phi} + \frac{\partial B_{z1}}{\partial z} = 0 \tag{3.2.4}$$

The inviscid velocity distribution will be used throughout equations (3.2.1) to (3.2.4).

In order to obtain a solution for the perturbation magnetic field equations the three field components will be separated now.

Differentiating equation (3.2.4) with respect to z and expressing $\frac{\partial^2 B_{\phi 1}}{\partial z \partial \phi}$ and $\frac{\partial^2}{\partial z \partial r}(rB_{r1})$ through equations (3.2.1) and (3.2.2) respectively, the partial differential equation defining B_{z1} can be written as

$$\frac{\partial^2 B_{z1}}{\partial r^2} + \frac{1}{r}\frac{\partial B_{z1}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 B_{z1}}{\partial \phi^2} + \frac{\partial^2 B_{z1}}{\partial z^2} = \nabla^2 B_{z1} = 0 \tag{3.2.5}$$

Next, the equation defining the magnetic field component $B_{\phi 1}$ will be derived. Differentiating equation (3.2.4) with respect to ϕ and eliminating $\frac{\partial^2}{\partial\phi\partial r}(rB_{r1})$ and $\frac{\partial^2 B_{z1}}{\partial\phi\partial z}$ through equations (3.2.3) and (3.2.1) respectively, the following equation is obtained for $B_{\phi 1}$:

$$\frac{\partial^2}{\partial r^2}(rB_{\phi 1}) + \frac{1}{r}\frac{\partial}{\partial r}(rB_{\phi 1}) + \frac{1}{r^2}\frac{\partial^2}{\partial\phi^2}(rB_{\phi 1}) + \frac{\partial^2}{\partial z^2}(rB_{\phi 1}) = \nabla^2(rB_{\phi 1}) =$$

$$\frac{1}{r}\frac{\partial}{\partial r} r^2 (E_{zo} + v_{ro}B_{\phi o} - v_{\phi o}B_{ro}) - r\frac{\partial E_{ro}}{\partial z}; \qquad (3.2.6)$$

Finally, an equation for B_{r1} is derived by differentiating (3.2.4) with respect to r and making the proper substitutions:

$$\frac{\partial^2 B_{r1}}{\partial r^2} + \frac{3}{r}\frac{\partial B_{r1}}{\partial r} + \frac{B_{r1}}{r^2} + \frac{1}{r^2}\frac{\partial^2 B_{r1}}{\partial\phi^2} + \frac{\partial^2 B_{r1}}{\partial z^2} =$$

$$= -\frac{2}{r}\frac{\partial B_{z1}}{\partial z} + \frac{\partial E_{\phi o}}{\partial z} - \frac{1}{r}\frac{\partial}{\partial\phi}(E_{zo} + v_{ro}B_{\phi o} - v_{\phi o}B_{ro}) ; \qquad (3.2.7)$$

Separating the variables, the solution for the equation (3.3.5) can be written as

$$B_{z1} = \sum_{\lambda,n} \Big\{\Big[A_\lambda \cos\lambda(\omega t - \phi) + B_\lambda \sin\lambda(\omega t - \phi)\Big] \cdot$$

$$\cdot \Big[A_n \cos nz + B_n \sin nz\Big] \cdot \qquad (3.2.8)$$

$$\cdot \Big[C_n I_\lambda(nr) + D_n K_\lambda(nr)\Big]\Big\}$$

An inspection of the zeroth order quantities which define the perturbation magnetic field (see equations (3.2.1) to (3.2.3) together with equations (2.2.4)) suggests the admissible values for λ and n as

$$\lambda = 1 \; ; \; n = ms \; ; \; s = 1, 3, 5, \ldots ; \; B_n \equiv 0 \quad (3.2.8a)$$

Hence

$$B_{z1} = \sqrt{\frac{2}{\pi}} \sum_s \cos(smr) \left\{ \left[a_{1s} \, I_1(smr) + b_{1s} \, K_1(smr) \right] \cos\,(\omega t - \phi) \right.$$

$$\left. + \left[c_{1s} \, I_1(smr) + d_{1s} \, K_1(smr) \right] \sin\,(\omega t - \phi) \right\} \qquad (3.2.9)$$

where the constants a_{1s}, b_{1s}, c_{1s}, d_{1s}, are left undefined.

Next, a solution will be found for equation (3.2.6) which can be rewritten in an expanded form as:

$$\nabla^2(rB_{\phi 1}) = 2(E_{zo} + v_{ro} \, B_{\phi o} - v_{\phi o} B_{ro}) +$$

$$+ r \frac{\partial}{\partial r} (E_{zo1} + v_{ro} B_{\phi o} - v_{\phi o} B_{ro}) =$$

$$= 2 \sqrt{\frac{2}{\pi}} \sum_s (sm) \sin\,(smr) \left\{ \left[A_s \, I_1(smr) + B_s \, K_1(smr) + \right. \right.$$

$$+ \left. \frac{1}{s^2 m} \sqrt{\frac{2}{\pi}} \left(\omega r + \frac{\rho_I V_I}{r} \right) \right] \cos\,(\omega t - \phi) + \left[C_s \, I_1(smr) + \right.$$

$$\left. \left. + D_s \, K_1(smr) - \frac{1}{s^2 m} \sqrt{\frac{2}{\pi}} \, \frac{Q}{r} \right] \sin\,(\omega t - \phi) \right\} \qquad (3.2.10)$$

Assume now a solution of the form

$$B_{\phi 1} = \frac{1}{r}\left[f_1(r,z,s)\cos(\omega t - \phi) + f_2(r,z,s)\sin(\omega t - \phi)\right] \tag{3.2.11}$$

Equation (3.2.10) yields two independent correlation for f_1 and f_2:

$$\frac{\partial^2 f_1}{\partial r^2} + \frac{1}{r}\frac{\partial f_1}{\partial r} - \frac{f_1}{r^2} + \frac{\partial^2 f_1}{\partial z^2} = 2\sqrt{\frac{2}{\pi}}\sum_s (sm)\sin(smr)\Big\{A_s I_1(smr) + B_o K_1(smr) + \frac{1}{s^2 m}\sqrt{\frac{2}{\pi}}\Big(\omega r + \frac{\rho_I V_I}{r}\Big)\Big\} \tag{3.2.12}$$

$$\frac{\partial^2 f_2}{\partial r^2} + \frac{1}{r}\frac{\partial f_2}{\partial r} - \frac{f_2}{r^2} + \frac{\partial^2 f_2}{\partial z^2} = 2\sqrt{\frac{2}{\pi}}\sum_s (sm)\sin(smr)\Big\{C_s I_1(smr) + D_s K_1(smr) - \frac{1}{s^2 m}\sqrt{\frac{2}{\pi}}\,\frac{Q}{r}\Big\} \tag{3.2.13}$$

The structure of (3.2.12) and (3.2.13) suggests the application of finite Fourier sine transforms to these equations. Indeed, introducing the transformation coordinates $r_1 = mr$; $z_1 = mz$; $m = \pi/L$ and defining the following transform functions:

$$\left.\begin{aligned} S_{1n}\left[f_1\right] &= \sqrt{\frac{2}{\pi}}\int_0^{\pi} f_1 \sin(nz_1)\,dz_1 \\ S_{2n}\left[f_2\right] &= \sqrt{\frac{2}{\pi}}\int_0^{\pi} f_2 \sin(nz_1)\,dz_1 \end{aligned}\right\} \tag{3.2.14}$$

The functions f_1 and f_2 are given by the inverse transforms:

$$\left.\begin{aligned} f_1 &= \sqrt{\frac{2}{\pi}} \sum_{n=1}^{\infty} S_{1n} \sin(nz_1) \\ f_2 &= \sqrt{\frac{2}{\pi}} \sum_{n=1}^{\infty} S_{1n} \sin(nz_1) \end{aligned}\right\} \qquad (3.2.15)$$

One may note here that

$$\int_0^{\pi} \sin(nz_1) \sin(sz_1)\, dz_1 = \begin{cases} 0 & \text{if } n \neq s \\ \pi/2 & \text{if } n = s \end{cases}$$

$$\text{and} \int_0^{\pi} g(r) \sin(nz_1)\, dz_1 = \left(\frac{1-(-1)^n}{n}\right) g(r)$$

$$n = 1, 2, 3, \ldots. \qquad (3.2.16)$$

Then for $n = 2, 4, 6, \ldots. 2k \quad k = 1, 2, 3, 4, \ldots.$

$$\frac{d^2S_{1n}}{dr_1^{\,2}} + \frac{1}{r_1}\frac{d_1S_{1n}}{dr_1} - (n^2 + \frac{1}{r_1^{\,2}})\, S_{1n} = n\sqrt{\frac{2}{\pi}}(f_1)_b\,((-1)^n-1) \equiv 0$$

$$\frac{d^2S_{2n}}{dr_1^{\,2}} + \frac{1}{r_1}\frac{dS_{2n}}{dr_1} - (n^2 + \frac{1}{r_1^{\,2}})\, S_{2n} = n\sqrt{\frac{2}{\pi}}(f_2)_b\,((-1)^n-1) \equiv 0$$

$$(3.2.17)$$

since $(-1)^n-1 \equiv 0$ for even values of n. $(f_1)_b$ and $(f_2)_b$ in (3.2.17) denote the values of f_1 and f_2 on the boundaries $z_1 = 0$;

$z_1 = \pi$ and they will be left undefined at the present.

The solutions to the homogeneous differential equations corresponding to (3.2.17) are given by

$$S_{1(2k)} = \sum_{k=1}^{\infty} \left[a_{2k} I_1 (2kmr) + b_{2k} K_1 (2kmr) \right]$$

(3.2.18)

$$S_{2(2k)} = \sum_{k=1}^{\infty} \left[c_{2k} I_1 (2kmr) + d_{2k} K_1 (2kmr) \right]$$

Since the zeroth order field quantities entering in the definition of the perturbation magnetic field do not contain terms with even values of the index n, the solutions $S_{1(2k)}$ and $S_{2(2k)}$ will be neglected again all together.

For odd values of n $\quad n \equiv s = 1, 3, 5, 7, \ldots.$ and $\left[(-1)^s - 1\right] = -2$ equations (3.2.12) and (3.2.13) are transformed into

$$\frac{d^2 S_{1s}}{dr_1^2} + \frac{1}{r_1} \frac{dS_{1s}}{dr_1} - (s^2 + \frac{1}{r_1^2}) S_{1s} = -2s \sqrt{\frac{2}{\pi}} (f_1)_b +$$

$$2 \frac{s}{m} \left[A_s I_1 (smr) + B_s K_1 (smr) + \frac{1}{s^2} \sqrt{\frac{2}{\pi}} (\frac{\omega}{m^2} r_1 + \frac{\rho_I V_I}{r_1}) \right]$$

(3.2.19)

$$\frac{d^2 S_{2s}}{dr_1^2} + \frac{1}{r_1} \frac{dS_{2s}}{dr_1} - (s^2 + \frac{1}{r_1^2}) S_{2s} = -2s \sqrt{\frac{2}{\pi}} (f_2)_b +$$

$$+ 2\frac{s}{m}\left[C_s\, I_1\,(smr) + D_s\, K_1\,(smr) - \frac{1}{s^2}\sqrt{\frac{2}{\pi}}\,\frac{Q}{r_1}\right] \qquad (3.2.20)$$

The general solutions for the above equations can be presented in the following form

$$S_{1s} = a_{2s}\, I_1\,(smr) + b_{2s}\, K_1\,(smr) + g_1(s,\, m,\, r) +$$

$$+ \frac{1}{m}\left[A_s\left(r_1 I_o(smr) - 2I_1(smr)\right) - B_s\, r_1\, K_o\,(smr)\right] - \frac{2}{s^3 m}\sqrt{\frac{2}{\pi}}\left(\frac{\omega}{m^2}\Gamma_1 + \frac{\rho_I V_I}{r_1}\right) \qquad (3.2.21)$$

$$S_{2s} = c_{2s}\, I_1\,(smr) + d_{2s}\, K_1\,(smr) + g_2(s,\, m,\, r) +$$

$$+ \frac{1}{m}\left[C_s\left(r_1 I_o(smr) - 2I_1(smr)\right) - D_s\, r_1\, K_o\,(smr)\right] + \frac{2}{s^3 m}\sqrt{\frac{2}{\pi}}\,\frac{Q}{r_1} \qquad (3.2.22)$$

In the equations (3.2.21) and (3.2.22) the constants a_{2s}, b_{2s}, c_{2s}, and d_{2s} are left undetermined just as are the functions g_1 and g_2 defining the particular integrals corresponding to the unknown functions $(f_1)_b$ and $(f_2)_b$, respectively.

Hence the general solution for $B_{\phi 1}$ is found to be

$$B_{\phi 1} = \frac{2}{\pi}\sum_s \sin\,(smz)\left\{\left[a_{2s}\frac{I_1(smr)}{r} + b_{2s}\frac{K_1(smr)}{r} +\right.\right.$$

$$+ \frac{g_1}{r} + A_s\left(I_o(smr) - \frac{2}{m}\frac{I_1(smr)}{r}\right) - B_s\, K_o\,(smr) -$$

$$- \frac{2}{s^3 m^2} \sqrt{\frac{2}{\pi}} \left(\omega + \frac{\rho_I V_I}{r^2} \right) \Bigg] \cos\,(\omega t - \phi) + \left[C_{2s} \frac{I_1(smr)}{r} + \right.$$

$$+ d_{2s} \frac{K_1(smr)}{r} + \frac{g_2}{r} + C_s \left(I_o(smr) - \frac{2}{m} \frac{I_1(smr)}{r} \right) - D_s K_o(smr) +$$

$$\left. + \frac{2}{s^3 m^2} \sqrt{\frac{2}{\pi}} \frac{Q}{r^2} \right] \sin\,(\omega t - \phi) \Big\}$$

$$s = 1, 3, 5, 7, \ldots . (2k + 1) \qquad (3.2.23)$$

Finally, the general solution of (3.2.7) defining the third perturbation field component B_{r1} will be found.

Equation (3.2.7) can be rewritten in expanded form:

$$\frac{\partial^2 B_{r1}}{\partial r^2} + \frac{3}{r} \frac{\partial B_{r1}}{\partial r} + \frac{B_{r1}}{r^2} + \frac{1}{r^2} \frac{\partial^2 B_{r1}}{\partial \phi^2} + \frac{\partial^2 B_{r1}}{\partial z^2} =$$

$$2\sqrt{\frac{2}{\pi}} \sum_s (sm) \sin(smz) \left\{ \left[a_{1s} \frac{I_1(smr)}{r} + b_{1s} \frac{K_1(smr)}{r} + \frac{1}{s^2 m} \sqrt{\frac{2}{\pi}} \frac{Q}{r^2} \right] \cos(\omega t - \phi) \right.$$

$$\left. + \left[c_{1s} \frac{I_1(smr)}{r} + d_{1s} \frac{K_1(smr)}{r} - \frac{1}{s^2 m} \sqrt{\frac{2}{\pi}} \left(\omega - \frac{\rho_I V_I}{r^2} \right) \right] \sin(\omega t - \phi) \right\}$$

$$(3.2.24)$$

Assume now a solution for B_{r1} of the following form:

$$B_{r1} = B_{r11}(r,z,s) \cos\,(\omega t - \phi) + B_{r12}(r,z,s) \sin\,(\omega t - \phi)$$

$$(3.3.25)$$

Substituting this back into (3.2.24) two independent equations are obtained for B_{r11} and B_{r12}:

$$\frac{\partial^2 B_{r11}}{\partial r^2} + \frac{3}{r}\frac{\partial B_{r11}}{\partial r} + \frac{\partial^2 B_{r11}}{\partial z^2} =$$

$$= 2\sqrt{\frac{2}{\pi}} \sum_s (sm) \sin (smz)\left[a_{1s} \frac{I_1(smr)}{r} + b_{1s} \frac{K_1(smr)}{r} + \frac{1}{s^2 m} \sqrt{\frac{2}{\pi}} \frac{Q}{r^2}\right] \tag{3.2.26}$$

$$\frac{\partial^2 B_{r12}}{\partial r^2} + \frac{3}{r}\frac{\partial B_{r12}}{\partial r} + \frac{\partial^2 B_{r12}}{\partial z^2} =$$

$$= 2\sqrt{\frac{2}{\pi}} \sum_s (sm) \sin (smz)\left[c_{1s} \frac{I_1(smr)}{r} + d_{1s} \frac{K_1(smr)}{r} - \frac{1}{s^2 m} \sqrt{\frac{2}{\pi}}\left(\omega - \frac{\rho_I V_I}{r^2}\right)\right] \tag{3.2.27}$$

Defining now the finite Fourier sine transforms of B_{r11} and B_{r12} as

$$S_{3n}\left[B_{r11}\right] = \sqrt{\frac{2}{\pi}} \int_0^{\pi} B_{r11} \sin (nz_1)\, dz_1$$

$$S_{4n}\left[B_{r12}\right] = \sqrt{\frac{2}{\pi}} \int_0^{\pi} B_{r12} \sin (nz_1)\, dz_1 \tag{3.2.28}$$

where $z_1 = mz$; $m = \pi/L$; $r_1 = mr$.

The inverse transforms yield:

$$B_{r11} = \sqrt{\frac{2}{\pi}} \sum_{n=1}^{\infty} S_{3n} \sin (nz_1)$$

$$B_{r12} = \sqrt{\frac{2}{\pi}} \sum_{n=1}^{\infty} S_{4n} \sin(nz_1) \qquad (3.2.24)$$

The differential equations (3.2.26), (3.2.27) can be transformed into

$$\frac{d^2S_{3n}}{dr_1^2} + \frac{3}{r_1}\frac{dS_{3n}}{dr_1} - n^2 S_{3n} = 0$$

$$(3.2.30)$$

$$\frac{d^2S_{4n}}{dr_1^2} + \frac{3}{r_1}\frac{dS_{4n}}{dr_1} - n^2 S_{4n} = 0$$

for $n = 2, 4, 6, 8, \ldots\; 2k$

$k = 1, 2, 3, 4$

and

$$\frac{d^2S_{3s}}{dr_1^2} + \frac{3}{r_1}\frac{dS_{3s}}{dr_1} - s^2S_{3s} = -2s\sqrt{\frac{2}{\pi}}(B_{r11})_b + \frac{2}{s}\sqrt{\frac{2}{\pi}}\frac{Q}{r_1^2} +$$

$$+2s\left(a_{1s}\frac{I_1(smr)}{r_1} + b_{1s}\frac{K_1(smr)}{r_1}\right)$$

$$(3.2.31)$$

$$\frac{d^2S_{4s}}{dr_1^2} + \frac{3}{r_1}\frac{dS_{4s}}{dr_1} - s^2S_{4s} = -2s\sqrt{\frac{2}{\pi}}(B_{r12})_b - \frac{2}{s}\sqrt{\frac{2}{\pi}}\left(\frac{\omega}{m^2} - \frac{\rho_I V_I}{r_1^2}\right) +$$

$$+2s\left(c_{1s}\frac{I_1(smr)}{r_1} + d_{1s}\frac{K_1(smr)}{r_1}\right)$$

for $n = s = 1, 3, 5, \ldots\; (2k + 1)$ $\qquad (3.2.32)$

where $(B_{r11})_b$ and $(B_{r12})_b$ are the values of B_{r11} and B_{r12} on the boundaries $z_1 = 0$ and $z_1 = \pi$. Both these functions will be left undefined at the present.

On the basis of an argument similar to that presented in connection with $S_{1(2k)}$ and $S_{2(2k)}$ all solutions S_{3n} and S_{4n} for even numbers n: $n = 2, 4, 6, 8, \ldots (2k)$ will be neglected again.

The differential equations (3.2.31) and (3.2.32) are satisfied by the general solutions:

$$S_{3s} = a_{3s} \frac{I_1(smr)}{r_1} + b_{3s} \frac{K_1(smr)}{r_1} + g_3(s, r_1) -$$

$$- \frac{2}{s^3}\sqrt{\frac{2}{\pi}} \frac{Q}{r_1^2} + a_{1s} I_o(smr) - b_{1s} K_o(smr) \quad (3.2.33)$$

$$S_{4s} = c_{3s} \frac{I_1(smr)}{r_1} + d_{3s} \frac{K_1(smr)}{r_1} + g_4(s, r_1) +$$

$$+ \frac{2}{s^3}\sqrt{\frac{2}{\pi}} \left(\frac{\omega}{m^2} - \frac{\rho_I V_I}{r_1^2}\right) + c_{1s} I_o(smr) - d_{1s} K_o(smr) \quad (3.2.34)$$

where the constants of integrations a_{3s}, b_{3s}, c_{3s}, d_{3s} will be left undetermined just as are the functions g_3 and g_4, representing the particular integrals corresponding to $(B_{r11})_b$ and $(B_{r12})_b$ in (3.2.31) and (3.2.32).

Thus the complete general solution for B_{r1} can be written as

$$B_{r1} = \sqrt{\frac{2}{\pi}} \sum_s \sin(smz) \left\{ \left[\frac{a_{3s}}{m} \frac{I_1(smr)}{r} + \frac{b_{3s}}{m} \frac{K_1(smr)}{r} + g_3 - \right.\right.$$

$$\left. - \frac{2}{s^3 m^2} \sqrt{\frac{2}{\pi}} \frac{Q}{r^2} + a_{1s} I_o(smr) - b_{1s} K_o(smr) \right] \cos(\omega t - \phi) +$$

$$+ \left[\frac{c_{3s}}{m} \frac{I_1(smr)}{r} + \frac{d_{3s}}{m} \frac{K_1(smr)}{r} + g_4 + \frac{2}{s^3 m^2} \sqrt{\frac{2}{\pi}} \left(\omega - \frac{\rho_I V_I}{r^2} \right) + \right.$$

$$\left.\left. + c_{1s} I_o(smr) - d_{1s} K_o(smr) \right] \sin(\omega t - \phi) \right\} \qquad (3.2.35)$$

$s = 1, 3, 5, 7, 9, \ldots. (2k + 1)$

The next step is to find those values of the integration constants a_{1s}, a_{2s}, a_{3s}, b_{1s}, b_{2s}, b_{3s}, c_{1s}, c_{2s}, c_{3s}, d_{1s}, d_{2s}, d_{3s}, for which the basic correlations (3.2.1) to (3.2.4) will be satisfied. The undefined particular integrals g_1, g_2, g_3 and g_4 will be determined in the same manner.

Substituting $B_{\phi 1}$ and B_{z1} given by (3.2.23) and (3.2.9) into (3.2.1) the following correlations are obtained:

$$\left.\begin{aligned} a_{1s} - sm\, c_{2s} &= (1 - 2s)\, C_{1s} \\ b_{1s} - sm\, d_{2s} &= D_s \\ c_{1s} + sm\, a_{2s} &= -(1 - 2s)\, A_s \\ d_{1s} + sm\, b_{2s} &= -B_s \end{aligned}\right\} \qquad (3.2.36)$$

$$\left.\begin{aligned} g_1 &\equiv 0 \\ g_2 &\equiv 0 \end{aligned}\right\} \qquad (3.2.36a)$$

Equation (3.2.2) with B_{r1} and B_{z1} given by (3.2.35) and (3.2.9) yields a second set of correlations:

$$\left.\begin{aligned} a_{1s} + sa_{3s} &= C_{1s} \\ b_{1s} + sb_{3s} &= D_s \\ c_{1s} + sc_{3s} &= -A_s \\ d_{1s} + sd_{3s} &= -B_s \end{aligned}\right\} \quad (3.2.37)$$

$$\left.\begin{aligned} g_3 &\equiv 0 \\ g_4 &\equiv 0 \end{aligned}\right\} \quad (3.2.37a)$$

The correlations obtained by substituting B_{r1} and $B_{\phi 1}$ into (3.2.3) are as follows:

$$\left.\begin{aligned} ma_{2s} - c_{3s} &= 2A_s \\ mb_{2s} - d_{3s} &= 0 \\ mc_{2s} + a_{3s} &= 2C_s \\ md_{2s} - b_{3s} &= 0 \end{aligned}\right\} \quad (3.2.38)$$

The divergence equation (3.2.4) does not yield new information, it merely reproduces some of the correlations given above.

Thus twelve equations (3.2.36, 3.2.37, 3.2.38) with twelve unknowns are obtained. Four of these correlations, however, are not independent (equations 3.2.38 for example can be obtained by combining equations 3.2.36 and 3.2.37). Hence the specification of an additional condition becomes necessary which must be based upon the physical nature of the field configuration discussed here and which yields four additional correlations similar to those presented above.

There are various possibilities for specifying the additional condition needed but not all of them are of practical value if the consistency of the solution is to be preserved.

The choice of an additional condition is governed by the assumption used in the solution of the perturbation velocity field equations, that is by the neglect of the end plate effects. Indeed, if the influence of end plates is disregarded the perturbation field component B_{z1} vanishes.

But $$B_{z1} = 0 \tag{3.2.39}$$

means $$a_{1s} = b_{1s} = c_{1s} = d_{1s} \equiv 0 \tag{3.2.40}$$

and the system of equations (3.2.36) to (3.2.38) yields the values of the remaining integration constants at once:

$$\left.\begin{array}{llll|llll} a_{2s} & = & \dfrac{2s-1}{sm} & A_s & a_{3s} & = & \dfrac{1}{s} & C_s \\ b_{2s} & = & \dfrac{-1}{sm} & B_s & b_{3s} & = & \dfrac{1}{s} & D_s \\ c_{2s} & = & \dfrac{2s-1}{sm} & C_s & c_{3s} & = & \dfrac{-1}{s} & A_s \\ d_{2s} & = & \dfrac{-1}{sm} & D_s & d_{3s} & = & \dfrac{-1}{s} & B_s \end{array}\right\} \quad (3.2.41)$$

The complete solutions for the first order perturbation magnetic field components can be written now as follows:

$$B_{r1} = \sqrt{\frac{2}{\pi}} \sum_s \sin(smz) \left\{\left[\frac{C_s}{sm} \frac{I_1(smr)}{r} + \frac{D_s}{sm} \frac{K_1(smr)}{r} - \right.\right.$$

$$\left. - \frac{2}{s^3m^2} \sqrt{\frac{2}{\pi}} \frac{Q}{r^2} \right] \cos(\omega t - \phi) + \left[- \frac{A_s}{sm} \frac{I_1(smr)}{r} - \frac{B_s}{sm} \frac{K_1(smr)}{r} + \right.$$

$$\left.\left. + \frac{2}{s^3m^2} \sqrt{\frac{2}{\pi}} (\omega - \frac{\rho_I V_I}{r^2})\right] \sin(\omega t - \phi)\right\} \quad (3.2.42)$$

$$B_{\phi 1} = \sqrt{\frac{2}{\pi}} \sum_s \sin(smz) \left\{\left[A_s(I_o(smz) - \frac{I_1(smr)}{smr}) - B_s(K_o(smr) + \frac{K_1(smr)}{smr})\right.\right.$$

$$\left. - \frac{2}{s^3m^2} \sqrt{\frac{2}{\pi}} (\omega + \frac{\rho_I V_I}{r^2})\right] \cos(\omega + \phi) + \left[C_s(I_o(smr) - \frac{I_1(smr)}{smr}) - D_s(K_o(smr)\right.$$

$$\left.\left. + \frac{K_1(smr)}{smr}) + \sqrt{\frac{2}{\pi}} \frac{Q}{r^2} \right] \sin(\omega t - \phi)\right\} \quad (3.2.43)$$

$$B_{z1} = 0 \; ; \quad (3.2.44)$$

where $s = 1, 3, 5, \ldots. (2k + 1)$;

the constants A_s, B_s, C_s and D_s are given by equations (2.2.41) and (2.2.42).

3) The First Order Electric Field

Since the known quantities entering in the definition of the first order electric field (such as the perturbation magnetic field components) were computed by neglecting the end plate effects, the same assumption will be used in the determination of the perturbation electric field.

As a result of this approximation no electrostatic field appears in the first order solution. Indeed, the electrostatic field denoted in the zeroth order solution as $\vec{E}_{o2}$ was the result of charge accumulation build up at the end plates in such a manner that the normal current component vanished at the nonconducting boundaries.

In the first order approximation the currents are unrestricted in the z-direction hence the charge accumulation with the corresponding electrostatic field are omitted from consideration. The complete first order electric field is defined by the equations:

$$\frac{1}{r}\frac{\partial E_{z1}}{\partial \phi} - \frac{\partial E_{\phi 1}}{\partial z} = -\frac{\partial B_{r1}}{\partial t} \tag{3.3.1}$$

$$\frac{\partial E_{r1}}{\partial z} - \frac{\partial E_{z1}}{\partial r} = -\frac{\partial B_{\phi 1}}{\partial t} \tag{3.3.2}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rE_{\phi 1}) - \frac{1}{r}\frac{\partial E_{r1}}{\partial \phi} = -\frac{\partial B_{z1}}{\partial t} \tag{3.3.3}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rE_{r1}) + \frac{1}{r}\frac{\partial E_{\phi 1}}{\partial \phi} + \frac{\partial E_{z1}}{\partial z} = 0 \qquad (3.3.4)$$

The assumption of vanishing divergence (equ. 3.3.4) is based now on the fact that with the removal of the end plates there is no charge accumulation in the entire flow region.

Following the method applied in the solution of the first order magnetic field, three separate second order partial differential equations can be derived for the three perturbation electric field components and the general solutions to these equations can be obtained by applying a finite Fourier sine transform to the equation defining E_{z1}, and finite Fourier cosine transforms to the equations defining the field components E_{r1} and $E_{\phi 1}$. A number of correlations is obtained then among the various integration constants by substituting the general solutions in the set of equations (3.3.1) to (3.3.4). If one specifies now an additional condition (the neglect of end plate effects in the given case), all integration constants can be determined uniquely.

The procedure outlined here has been in fact carried out but it will not be presented here for brevity.

Instead, a solution for the perturbation electric field will be obtained by neglecting the end plate effects a priori, and the simplified set of differential equations corresponding to this field configuration will be solved. (The two procedures outlined above are consistent as is evidenced by the fact that the solutions were found to be identical.)

Indeed, the radial and azimuthal components of the electric field vanish in the absence of the end plates and the various field components cease to be functions of the z coordinate.

Thus

$$E_{r1} = E_{\phi 1} = \frac{\partial}{\partial z} \equiv 0 \qquad (3.3.5)$$

and the equations (3.3.1) to (3.3.4) can be rewritten as

$$\left.\begin{aligned} \frac{\partial E_{z1}}{\partial \phi} &= - r \frac{\partial B_{r1}}{\partial t} \\ \frac{\partial E_{z1}}{\partial r} &= \frac{\partial B_{\phi 1}}{\partial t} \end{aligned}\right\} \qquad (3.3.6)$$

In this approximation one should not attempt to satisfy the divergence equation (3.3.4) for the following reason. The neglect of end plate effects implies that the physical quantities do not vary in the z-direction. However, as the result of the finite Fourier transforms applied during the previous derivations with respect to the z-coordinate, all first order field quantities are expressed in infinite sine or cosine series containing z as argument. Thus the condition $\frac{\partial E_{z1}}{\partial z} = 0$ cannot be satisfied (A similar situation exists when one expresses the unit step function through a Fourier series of argument x. Although the function itself is constant, its derivative with respect to x will not vanish though it may numerically approach zero if a sufficiently large number of terms of the Fourier series is taken.)

Equations (3.3.6) can be rewritten in expanded form as:

$$\frac{\partial E_{z1}}{\partial \phi} = \omega \sqrt{\frac{2}{\pi}} \sum_s \sin (smz)\Big\{\Big[\frac{A_s}{sm} I_1 (smr) + \frac{B_s}{sm} K_1 (smr) - $$

$$- \frac{2}{s^3 m^2}\sqrt{\frac{2}{\pi}} (\omega r - \frac{\rho_I V_I}{r})\Big] \cos (\omega t - \phi) + \Big[\frac{C_s}{sm} I_1 (smr) +$$

$$+ \frac{D_s}{sm} K_1 (smr) - \frac{2}{s^3 m^2}\sqrt{\frac{2}{\pi}}\frac{Q}{r}\Big] \sin (\omega t - \phi)\Big\} \quad (3.3.7a)$$

$$\frac{\partial E_{z1}}{\partial r} = \omega \sqrt{\frac{2}{\pi}} \sum_s \sin (smz)\Big\{\Big[C_s \, (I_o(smr) - \frac{I_1(smr)}{smr}) -$$

$$- D_s \, (K_o(smr) - \frac{K_1(smr)}{smr}) + \frac{2}{s^3 m^2}\sqrt{\frac{2}{\pi}}\frac{Q}{r^2}\Big] \cos (\omega t - \phi) +$$

$$+\Big[-A_s(I_o(smr) - \frac{I_1(smr)}{smr}) + B_s(K_o(smr) + \frac{K_1(smr)}{smr}) +$$

$$+ \frac{2}{s^3 m^2}\sqrt{\frac{2}{\pi}} (\omega + \frac{\rho_I V_I}{r^2})\Big] \sin (\omega t - \phi)\Big\} \quad (3.3.7b)$$

The solution for E_{z1} defined by equations (3.3.7a) and (3.3.7b) can be written up at once:

$$E_{z1} = \omega\sqrt{\frac{2}{\pi}}\sum_s \sin(smz)\left\{\left[\frac{C_s}{sm} I_1(smr) + \right.\right.$$

$$+ \frac{D_s}{sm} K_1(smr) - \frac{2}{s^3 m^2}\sqrt{\frac{2}{\pi}}\,\frac{Q}{r}\Big]\cos(\omega t - \phi) +$$

$$+ \Big[- \frac{A_s}{sm} I_1(smr) - \frac{B_s}{sm} K_1(smr) + \frac{2}{s^3 m^2}\sqrt{\frac{2}{\pi}}\,(\omega r -$$

$$\left.\left. - \frac{\rho_I V_I}{r})\right]\sin(\omega t - \phi)\right\} \qquad (3.3.8)$$

Equation (3.3.8) together with $E_{r1} = E_{\phi 1} = 0$ defines the perturbation electric field.

PART B. ENERGY CONSIDERATIONS

IV. THE POWER GENERATION

1. The Energy Equation

The complete energy equation for an electrically conducting fluid moving in the presence of a magnetic field can be written as (see references [23] and [24]).

$$\rho \frac{D}{Dt}\left(C_p T + \frac{v^2}{2}\right) = \nabla \cdot (K \nabla T) + \vec{v} \cdot (\vec{I} \times \vec{B}) + \\ + \frac{I^2}{\sigma} + \rho\nu\vec{v} \cdot (\nabla^2 \vec{v}) + \Phi \quad (4.1.1)$$

where C_p is the specific heat of the medium at constant pressure,

T is the temperature,

K is the thermal conductivity of the medium, and

Φ is defined as the hydrodynamic dissipation function;

$$\Phi = \rho\nu\left[\left(\frac{\partial v_\ell}{\partial x_k} + \frac{\partial v_k}{\partial x_\ell}\right)^2 + 2\left(\frac{\partial v_k}{\partial x_k}\right)^2 - \frac{2}{3}(\nabla \cdot \vec{v})^2\right]$$

$$k = 1, 2, 3 \; ; \; \ell = 1, 2, 3 \quad (4.1.2)$$

(In expression 4.1.2 the summation convention is used).

Equation (4.1.1) introduces another field variable: the temperature. If one succeeds to compute the temperature change characterizing the generator-cycle, the change of energy content per unit mass of the fluid can be determined subsequently. This in turn yields the necessary information about the power extracted from the fluid in form of electrical energy.

Although equation (4.1.1) is linear in T its solution is quite difficult for the given case, due to the complicated expressions found for the electromagnetic field components.

The nature of the process discussed here, however, makes it possible to determine the power density generated in the fluid without computing the temperature field distribution explicitly.

Indeed, the high electrical resistivity of the fluid makes the ohmic losses dominant over the losses due to viscous effects and thermal conduction. The generator chamber is assumed to have perfect thermal insulation hence there are no heat losses to the environment. Furthermore, the power extracted from the fluid in magnetohydrodynamic generators is usually of the same order of magnitude as the heat gained by the medium through the ohmic heating because of its low electrical conductivity. Hence there will be no large temperature gradients induced by the generator cycle.

Under these conditions the energy balance of the working fluid can be described by forming the scalar product of the velocity with the momentum equation:

$$\vec{v} \cdot \frac{D\vec{v}}{Dt} = -\frac{1}{\rho}\vec{v} \cdot \nabla p + \frac{1}{\rho}\vec{v} \cdot (\vec{I} \times \vec{B}) + \nu\vec{v} \cdot \nabla^2\vec{v}$$

or neglecting the viscous term:

$$\underset{\text{(I)}}{\vec{v} \cdot \frac{D\vec{v}}{Dt}} = \underset{\text{(II)}}{-\frac{1}{\rho}\vec{v} \cdot \nabla p} + \underset{\text{(III)}}{\frac{1}{\rho}\vec{v} \cdot (\vec{I} \times \vec{B})} \qquad (4.1.3)$$

where

(I) is the rate of change of the kinetic energy of the fluid (positive if the velocity increases)

(II) is the rate of work done by the pressure forces on the fluid (positive in case of adverse pressure gradient)

(III) is the rate of work done by the electromagnetic forces on the fluid (positive for electromagnetic driving i.e. pumping action).

For steady, incompressible flow equation (4.1.3) can be rewritten as

$$\vec{v} \cdot (\vec{I} \times \vec{B}) = \vec{v} \cdot \left(\rho \frac{D\vec{v}}{Dt} + \nabla p\right) =$$

$$= \frac{D}{Dt} \left(\frac{\rho}{2} v^2 + p\right) \qquad (4.1.4)$$

In case of generator action the power is extracted from the fluid, thus $\vec{v} \cdot (\vec{I} \times \vec{B})$ is a negative quantity. Hence the power

$$P_v = -\vec{v} \cdot (\vec{I} \times \vec{B}) = -\frac{Dp_T}{Dt} \qquad (4.1.5)$$

where the subscript v indicates quantities per unit volume and p_T is the total pressure defined as

$$p_T = \frac{1}{2} \rho v^2 + p$$

As could have been anticipated, the generated power density is proportional to the total pressure drop.

The power density given by (4.1.5) is not all useful power as can be seen from the following argument:

$$
\begin{aligned}
-\vec{v}\cdot(\vec{I}\times\vec{B}) &= \vec{I}\cdot(\vec{v}\times\vec{B}) = \\
&= \vec{I}\cdot\left(-\vec{E}+\frac{\vec{I}}{\sigma}\right) = \\
&= -\vec{E}\cdot\vec{I}+\frac{I^2}{\sigma} \qquad (4.1.6)
\end{aligned}
$$

The useful or effective power generated is given by the $(-\vec{E}\cdot\vec{I})$ term (the minus sign indicates that the energy is being extracted from the fluid), and (I^2/σ) is the ohmic dissipation.

The basic performance characteristics of the converter system can be defined now as

$$
\eta \equiv \frac{\text{effective power extracted}}{\text{rate of work done by the fluid}} = \frac{P_e}{P} \qquad (4.1.7)
$$

where

$$
\left.
\begin{aligned}
P_e &= \int_v (-\vec{E}\cdot\vec{I})\, dv \\
P &= \int_v \left[-\vec{v}\cdot(\vec{I}\times\vec{B})\right] dv
\end{aligned}
\right| \qquad (4.1.8)
$$

The integrals are taken over the volume of the generator chamber.

The total ohmic losses are given by the integral

$$L = \int_v \frac{I^2}{\sigma} \, dv \tag{4.1.8a}$$

The electrical efficiency defined by equation (4.1.7) is closely related to the magnitude of the "slip" s, defined in electrical engineering as

$$s = \frac{v_{ph} - v_c}{v_{ph}} \tag{4.1.9}$$

where

v_c is the conductor velocity and

v_{ph} is the phase velocity (velocity of propagation of the magnetic field).

For the case of slug motion of a conducting fluid between two infinite parallel plates in a transverse magnetic field propagating in the direction of fluid motion the following relation has been obtained by I. Bernstein (see ref. [12]):

$$\eta = \frac{1}{1 - s} \tag{4.1.9a}$$

Hence under idealized conditions (uniform motion, zero slip, absence of eddy currents) the electrical efficiency may approach to unit value.

In the following, dimensionless quantities will be used again and all variables entering in the definition of the various power densities will be expanded in terms of the magnetic Reynolds number.

Hence

$$P_v = P_{vo} + R_m P_{v1} + \dots\dots =$$

$$= -(\vec{v}_o + R_m \vec{v}_1 + \dots\dots) \cdot \Big[(\vec{I}_o + R_m \vec{I}_1 + \dots\dots) \times$$

$$\times (\vec{B}_o + R_m \vec{B}_1 + \dots\dots)\Big] \qquad (4.1.10)$$

or since

$$\vec{I}_o = \vec{E}_o + \vec{v}_o \times \vec{B}_o \quad \text{and}$$

$$\vec{I}_1 = \vec{E}_1 + \vec{v}_1 \times \vec{B}_o + \vec{v}_o \times \vec{B}_1$$

the following expressions can be obtained:

$$P_{vo} = -\vec{v}_o \cdot (\vec{I}_o \times \vec{B}_o) =$$

$$= -\vec{v}_o \cdot \Big[\vec{E}_o \times \vec{B}_o + (\vec{v}_o \times \vec{B}_o) \times \vec{B}_o\Big] \qquad (4.1.11)$$

$$P_{v1} = -\Big[\vec{v}_1 \cdot (\vec{I}_o \times \vec{B}_o) + \vec{v}_o \cdot (\vec{I}_1 \times \vec{B}_o) +$$

$$+ \vec{v}_o \cdot (\vec{I}_o \times \vec{B}_1)\Big] =$$

$$= -\vec{v}_1 \cdot \Big[\vec{E}_o \times \vec{B}_o + (\vec{v}_o \times \vec{B}_o) \times \vec{B}_o\Big] +$$

$$-\vec{v}_o \cdot \Big[\vec{E}_o \times \vec{B}_1 + \vec{E}_1 \times \vec{B}_o + (\vec{v}_o \times \vec{B}_o) \times \vec{B}_1 +$$

$$+ (\vec{v}_o \times \vec{B}_1) \times \vec{B}_o + (\vec{v}_1 \times \vec{B}_o) \times \vec{B}_o\Big] \qquad (4.1.12)$$

The ohmic losses are represented by the term $\vec{I} \cdot \vec{I}$. Applying the series expansion technique (ohmic losses $\equiv L_v$):

$$L_v = L_{vo} + R_m L_{v1} + \ldots. =$$
$$= (\vec{I}_o + R_m \vec{I}_1 + \ldots.) \cdot (\vec{I}_o + R_m \vec{I}_1 + \ldots.) =$$
$$= \left[(\vec{E}_o + \vec{v}_o \times \vec{B}_o) + R_m (\vec{E}_1 + \vec{v}_o \times \vec{B}_1 + \vec{v}_1 \times \vec{B}_o) + \ldots.\right]$$
$$\cdot \left[(\vec{E}_o + \vec{v}_o \times \vec{B}_o) + R_m (\vec{E}_1 + \vec{v}_o \times \vec{B}_1 + \vec{v}_1 \times \vec{B}_o) + \ldots.\right] \tag{4.1.13}$$

Equating the terms containing like powers of R_m:

$$L_{vo} = \vec{I}_o \cdot \vec{I}_o =$$
$$= (\vec{E}_o + \vec{v}_o \times \vec{B}_o) \cdot (\vec{E}_o + \vec{v}_o \times \vec{B}_o) \tag{4.1.14}$$

$$L_{v1} = 2\, \vec{I}_o \cdot \vec{I}_1 =$$
$$= 2\, (\vec{E}_o + \vec{v}_o \times \vec{B}_o) \cdot (\vec{E}_1 + \vec{v}_o \times \vec{B}_1 + \vec{v}_1 \times \vec{B}_o) \tag{4.1.15}$$

Finally the effective power is defined as $P_{ev} = -\vec{E} \cdot \vec{I}$. Expanding the terms in infinite series

$$P_{ev} = P_{evo} + R_m P_{ev1} + \dots =$$

$$= -(\vec{E}_o + R_m \vec{E}_1 + \dots) \cdot (\vec{I}_o + R_m \vec{I}_1 + \dots) =$$

$$= -(\vec{E}_o + R_m \vec{E}_1 + \dots) \cdot \Big[(\vec{E}_o + \vec{v}_o \times \vec{B}_o) +$$

$$+ R_m(\vec{E}_1 + \vec{v}_o \times \vec{B}_1 + \vec{v}_1 \times \vec{B}_o) + \dots \Big] \qquad (4.1.16)$$

Thus the following correlations are obtained:

$$P_{evo} = -\vec{E}_o \cdot \vec{I}_o = -\vec{E}_o \cdot (\vec{E}_o + \vec{v}_o \times \vec{B}_o) \qquad (4.1.17)$$

$$P_{ev1} = -(\vec{E}_o \cdot \vec{I}_1 + \vec{E}_1 \cdot \vec{I}_o) =$$

$$= -\Big[\vec{E}_o \cdot (\vec{E}_1 + \vec{v}_o \times \vec{B}_1 + \vec{v}_1 \times \vec{B}_o) +$$

$$+ \vec{E}_1 \cdot (\vec{E}_o + \vec{v}_o \times \vec{B}_o)\Big] \qquad (4.1.18)$$

In order to compute the electrical efficiency of the generator cycle (η) the above expressions must be integrated over the total volume of the generator chamber.

2. Zeroth Order Power

Expanding equ. (4.1.10) and making the proper substitutions the following expression is obtained for the zeroth order power density:

$$P_{vo} = -(v_{\phi o} B_{ro} - v_{ro} B_{\phi o}) I_{zo} =$$

$$= -(\rho_I V_I \cos(\omega t - \phi) - Q \sin(\omega t - \phi))\sqrt{\frac{2}{\pi}} \sum_s$$

$$\sum_s (sm) \sin(smz) \left\{\left[A_s \frac{I_1(smr)}{r} + \right.\right.$$

$$\left. + B_s \frac{K_1(smr)}{r}\right] \cos(\omega t - \phi) + \left[C_s \frac{I_1(smr)}{r} + \right.$$

$$\left.\left. + D_s \frac{K_1(smr)}{r}\right] \sin(\omega t - \phi)\right\} =$$

$$= -\sqrt{\frac{2}{\pi}} \sum_s (sm) \sin(smz) \left\{\rho_I V_I (A_s \frac{I_1}{r} + B_s \frac{K_1}{r}) \cos^2(\omega t - \phi) - \right.$$

$$- Q (C_s \frac{I_1}{r} + D_s \frac{K_1}{r}) \sin^2(\omega t - \phi) +$$

$$+ \frac{1}{2} \left[\rho_I V_I (C_s \frac{I_1}{r} + D_s \frac{K_1}{r}) - Q (A_s \frac{I_1}{r} + \right.$$

$$\left.\left. + B_s \frac{K_1}{r})\right] \sin 2(\omega t - \phi)\right\} \qquad (4.2.1)$$

where the following symbols are introduced:

$$I_1 \equiv I_1(smr)$$

$$K_1 \equiv K_1(smr)$$

and as earlier $\qquad s = 1, 3, 5, 7, \ldots. (2k + 1)$

The total power generated in the generator chamber is given by

$$P_o = \int_0^L \int_0^{2\pi} \int_{\rho_o}^{\rho_I} P_{vo}\, r\, dr\, d\phi\, dz \qquad (4.2.2)$$

Since

$$\int_0^L (sm) \sin (smz)\, dz = -\cos \left(s \frac{\pi}{L} z\right) \Big/_0^L = 2$$

$$\int_0^{2\pi} \cos^2 (\omega t - \phi) d\phi = \int_0^{2\pi} \sin^2 (\omega t - \phi)\, d\phi = \pi$$

$$\int_0^{2\pi} \sin 2 (\omega t - \phi)\, d\phi \equiv 0$$

$$\int_{\rho_o}^{\rho_I} (I_1 + K_1)\, dr = \frac{1}{sm} \left| I_o - K_o \right|_{\rho_o}^{\rho_I}$$

where

$$I_o \equiv I_o\,(smr)$$

$$K_o \equiv K_o\,(smr)$$

The integral (4.2.2) yields the following expression for the zeroth order total power:

$$P_o = -\frac{2\pi}{m} \sqrt{\frac{2}{\pi}} \sum_s \frac{1}{s} \Big[(\rho_I V_I A_s - QC_s)($$

$$(I_o(sm\rho_I) - I_o(sm\rho_o)) + (\rho_I V_I B_s - Q\, D_s)($$

$$(K_o(sm\rho_o) - K_o\,(sm\rho_I))\Big] \qquad (4.2.3)$$

Next, the total ohmic losses shall be computed. As given by equation (4.1.13)

$$L_{vo} = I_{ro}^{\ 2} + I_{\phi o}^{\ 2} + I_{zo}^{\ 2} \qquad (4.2.4)$$

and

$$L = \int_{o}^{L} \int_{o}^{2\pi} \int_{\rho_o}^{\rho_I} L_{vo}\ r\ dr\ d\phi\ dz \qquad (4.2.5)$$

One may note that

$$I_{ro} = E_{ro}$$

$$I_{\phi o} = E_{\phi o}$$

$$I_{zo} = E_{zo} + v_{ro} B_{\phi o} - v_{\phi o} B_{ro}$$

Furthermore;

$$\begin{aligned} I_{ro}^{\ 2} = & \frac{2}{\pi} \Big[\sum_s (sm) \cos (smz)\, F_{1s}{}^{(r)} \Big]^2 \cos^2 (\omega t - \phi) + \\ & + \frac{2}{\pi} \Big[\sum_s (sm) \cos (smz)\, F_{2s}{}^{(r)} \Big]^2 \sin^2 (\omega t - \phi) + \\ & + \frac{4}{\pi} \Big[\sum_s (sm) \cos (smz)\, F_{1s}{}^{(r)} \Big] \cos (\omega t - \phi) \ \cdot \\ & \cdot \Big[\sum_s (sm) \cos (smz)\, F_{2s}{}^{(r)} \Big] \sin (\omega t - \phi) \ ; \end{aligned} \qquad (4.2.6)$$

$$I_{\phi o}^{\;2} = \frac{2}{\pi}\left[\sum_s \cos(smz)\, F_{1s}^{(\phi)}\right]^2 \cos^2(\omega t - \phi) +$$

$$+ \frac{2}{\pi}\left[\sum_s \cos(smz)\, F_{2s}^{(\phi)}\right]^2 \sin^2(\omega t - \phi) +$$

$$+ \frac{4}{\pi}\left[\sum_s \cos(smz)\, F_{1s}^{(\phi)}\right]\left[\sum_s \cos(smz)\, F_{2s}^{(\phi)}\right]\sin(\omega t-\phi)\cos(\omega t-\phi) \qquad (4.2.7)$$

$$I_{zo}^{\;2} = \frac{2}{\pi}\left[\sum_s (sm)\sin(smz)\, F_{1s}^{(z)}\right]^2 \cos^2(\omega t - \phi) +$$

$$+ \frac{2}{\pi}\left[\sum_s (sm)\sin(smz)\, F_{2s}^{(z)}\right]^2 \sin^2(\omega t - \phi) +$$

$$+ \frac{4}{\pi}\left[\sum_s (sm)\sin(smz)\, F_{1s}^{(z)}\right] \cos(\omega t - \phi) \;\cdot$$

$$\cdot \left[\sum_s (sm)\sin(smz)\, F_{2s}^{(z)}\right] \sin(\omega t - \phi) \qquad (4.2.8)$$

where

$$F_{1s}^{(r)} \equiv A_s\left(-I_o + \frac{I_1}{smr}\right) + B_s\left(K_o + \frac{K_1}{smr}\right) + \frac{2}{s^3m^2}\,\frac{2}{\pi}\left(\omega + \frac{\rho_I V_I}{r^2}\right)$$

$$F_{2s}^{(r)} \equiv C_s\left(-I_o + \frac{I_1}{smr}\right) + D_s\left(K_o + \frac{K_1}{smr}\right) - \frac{2}{s^3m^2}\,\frac{2}{\pi}\,\frac{Q}{r^2} \qquad (4.2.9)$$

$$F_{1s}^{(\phi)} \equiv C_s\frac{I_1}{r} + D_s\frac{K_1}{r} - \frac{2}{s^2m}\,\frac{2}{\pi}\,\frac{Q}{r^2}$$

$$F_{2s}^{(\phi)} \equiv -A_s\frac{I_1}{r} - B_s\frac{K_1}{r} + \frac{2}{s^2m}\,\frac{2}{\pi}\left(\omega - \frac{\rho_I V_I}{r^2}\right) \qquad (4.2.10)$$

$$F_{1s}^{(z)} \equiv A_sI_1 + B_sK_1$$

$$F_{2s}^{(z)} \equiv C_sI_1 + D_sK_1 \qquad (4.2.11)$$

First of all, one notes that

$$\int_0^{2\pi} \sin(\omega t - \phi)\cos(\omega t - \phi)\, d\phi \equiv 0$$

hence the non-quadratic terms drop out from equations (4.2.6) to (4.2.8) after integrating with respect to ϕ.

Furthermore, expanding the quadratures as, for example:

$$\left[\sum_s (sm)\cos(smz) F_{1s}\right]^2 =$$

$$= m^2 \left[F_{11}^{\ 2} \cos^2 z_1 + 3^2 F_{13}^{\ 2} \cos^2 3z_1 + 5^2 F_{15} \cos^2 5z_1 + \ldots\right.$$

$$\left. + \ 2.3\, F_{11} F_{13} \cos z_1 \cos 3z_1 + 2.5\, F_{11} F_{15} \cos z_1 \cos 5z_1 + .\right]$$

where $z_1 \equiv mz$; $m = \frac{\pi}{L}$.

Since

$$\int_0^{\pi} \cos sz_1 \cos nz_1 \, dz_1 = \begin{cases} 0 & \text{for } s \neq n \\ \frac{\pi}{2} & \text{for } s = n \end{cases}$$

$$\int_0^{L} \left[\sum_s (sm)\cos(smz) F_{is}\right]^2 dz =$$

$$= \frac{\pi}{2}\, m \sum_s s^2 F_{is}^{\ 2} \quad ; \quad i = 1, 2, \qquad (4.4.12)$$

Therefore

$$\int_0^L \int_0^{2\pi} \int_{\rho_o}^{\rho_I} \left(I_{ro}^{\;2} + I_{\phi o}^{\;2} + I_{zo}^{\;2}\right) r\, dr\, d\phi\, dz \;=$$

$$= \frac{\pi}{m} \int_{\rho_o}^{\rho_I} \sum_s \Big[m^2 s^2 \, {F_{1s}^{(r)}}^2 + m^2 s^2 \, {F_{2s}^{(r)}}^2 + {F_{1s}^{(\phi)}}^2 +$$

$$+ \; {F_{2s}^{(\phi)}}^2 + m^2 s^2 \, {F_{1s}^{(z)}}^2 + m^2 s^2 \, {F_{2s}^{(z)}}^2 \Big] \, r\, dr \qquad (4.2.13)$$

Substituting (4.2.9) to (4.2.11) into (4.2.13) and performing the algebraic simplification available, the zeroth order ohmic losses can be computed as follows:

$$L_o = \frac{\pi}{m} \int_{\rho_o}^{\rho_I} \sum_s r \Big\{ \left(A_s^{\;2} + C_s^{\;2}\right) \Big[s^2 m^2 \left(I_o^{\;2} + I_1^{\;2}\right) +$$

$$+ \; \frac{2 I_1^{\;2}}{r^2} - 2sm \, \frac{I_o I_1}{r} \Big] + \left(B_s^{\;2} + D_s^{\;2}\right) \Big[s^2 m^2 \left(K_o^{\;2} + K_1^{\;2}\right) +$$

$$+ \; 2 \, \frac{K_1^{\;2}}{r^2} + 2sm \, \frac{K_o K_1}{r} \Big] + 2\left(A_s B_s + C_s D_s\right) \Big[s^2 m^2 \left(I_1 K_1 -\right.$$

$$\left. - \; I_o K_o\right) + \frac{sm}{r} \left(I_1 K_o - I_o K_1\right) + 2 \, \frac{I_1 K_1}{r^2} \Big] + \frac{16}{s^4 m^2 \pi} \Big(\omega^2 +$$

$$+ \; \frac{\rho_I^{\;2} V_I^{\;2} + Q^2}{r^4} \Big) - \frac{4\omega}{s} \sqrt{\frac{2}{\pi}} \left(A_s I_o - B_s K_o\right) + \frac{4}{s} \sqrt{\frac{2}{\pi}} \Big(Q C_s -$$

$$- \; \rho_I V_I A_s \Big) \left(\frac{I_o}{r^2} - \frac{2}{sm} \frac{I_1}{r^3} \right) - \frac{4}{s} \sqrt{\frac{2}{\pi}} \left(Q D_s - \rho_I V_I B_s\right) \left(\frac{K_o}{r^2} + \frac{2}{sm} \frac{K_1}{r^3} \right) \Big\} \, dr \;=$$

$$= \frac{\pi}{m} \sum_{s}^{\infty} \Big[smr \Big\{ (A_s^2 + C_s^2) I_o I_1 - (B_s^2 + D_s^2) K_o K_1 -$$

$$- 2 (A_s B_s + C_s D_s) I_1 K_o \Big\} - (A_s I_1 + B_s K_1)^2 -$$

$$- (C_s I_1 + D_s K_1)^2 + \frac{8}{s^4 m^2 \pi} \left(\omega r^2 - \frac{\rho_I^2 V_I^2 + Q^2}{r^2}\right) -$$

$$- \frac{4\omega}{s^2 m} \sqrt{\frac{2}{\pi}}\, r\, (A_s I_1 + B_s K_1) + \frac{4}{s^2 m} \sqrt{\frac{2}{\pi}} (QC_s -$$

$$- \rho_I V_I A_s) \frac{I_1}{r} + \frac{4}{s^2 m} \sqrt{\frac{2}{\pi}} (QD_s - \rho_I V_I B_s) \frac{K_1}{r} \Big]_{\rho_o}^{\rho_I} \qquad (4.2.14)$$

with $s = 1, 3, 5, 7, 9, \ldots . (2k + 1)$.

The coefficients A_s, B_s, C_s and D_s are defined in Sec. (II.2).

3. The First Order Power

The expression for the first order power production density can be written in expanded form as

$$P_{v1} = - \vec{v}_o \cdot (\vec{I}_o \times \vec{B}_1 + \vec{I}_1 \times \vec{B}_o) - \vec{v}_1 \cdot (\vec{I}_o \times \vec{B}_o) =$$

$$= - (E_{zo} + v_{ro} B_{\phi o} - v_{\phi o} B_{ro})(v_{\phi o} B_{r1} - v_{ro} B_{\phi 1} + v_{\phi 1} B_{ro} - v_{r1} B_{\phi o})$$

$$- (E_{z1} + v_{r1} B_{\phi o} - v_{\phi 1} B_{ro} + v_{ro} B_{\phi 1} - v_{\phi o} B_{r1})(v_{\phi o} B_{ro} - v_{ro} B_{\phi o})$$

$$(4.3.1)$$

The ohmic losses per unit volume are given by the following expression:

$$L_{v1} = 2\,\vec{I}_o \cdot \vec{I}_1 =$$

$$= 2\,(\vec{E}_{zo} + v_{ro}B_{\phi o} - v_{\phi o}B_{ro})\,(E_{z1} + v_{r1}B_{\phi o} -$$

$$- v_{\phi 1}B_{ro} + v_{ro}B_{\phi 1} - v_{\phi o}B_{r1}) \qquad (4.3.2)$$

Finally, the net power density generated can be written as

$$P_{ev1} = -\vec{E}_o \cdot \vec{I}_1 - \vec{E}_1 \cdot \vec{I}_o =$$

$$= -E_{zo}\,(E_{z1} + v_{r1}B_{\phi o} - v_{\phi 1}B_{ro} +$$

$$+ v_{ro}B_{\phi 1} - v_{\phi 1}B_{r1}) -$$

$$- E_{z1}\,(E_{zo} + v_{ro}B_{\phi o} - v_{\phi o}B_{ro}) \qquad (4.3.3)$$

The above expressions will be rewritten now in a somewhat different form:

$$P_{v1} = P_{v11} + P_{v12}$$

where

$$P_{v11} = (E_{zo} + v_{ro}B_{\phi o} - v_{\phi o}B_{ro})\,(v_{ro}B_{\phi 1} - v_{\phi o}B_{r1}) +$$

$$+ (E_{z1} + v_{ro}B_{\phi 1} - v_{\phi o}B_{r1})\,(v_{ro}B_{\phi o} - v_{\phi o}B_{ro})\ ;$$

$$P_{v12} = (E_{zo} + v_{ro}B_{\phi o} - v_{\phi o}B_{ro})(v_{r1}B_{\phi o} - v_{\phi 1}B_{ro}) +$$

$$+ (v_{r1}B_{\phi o} - v_{\phi 1}B_{ro})(v_{ro}B_{\phi o} - v_{\phi o}B_{ro})\ ; \qquad (4.3.4)$$

$$L_{v1} = L_{v11} + L_{v12}$$

where

$$L_{v11} = 2\,(E_{zo} + v_{ro}B_{\phi o} - v_{\phi o}B_{ro})\,(E_{z1} + v_{ro}B_{\phi 1} - v_{\phi o}B_{r1})\;;$$

$$L_{v12} = 2\,(E_{zo} + v_{ro}B_{\phi o} - v_{\phi o}B_{ro})\,(v_{r1}B_{\phi o} - v_{\phi 1}B_{ro}) \quad (4.3.5)$$

and finally

$$P_{ev1} = P_{ev11} + P_{ev12}$$

where

$$P_{ev11} = -2\,E_{zo}E_{z1} + E_{zo}(v_{\phi o}B_{r1} - v_{ro}B\phi 1)$$

$$+ \; E_{z1}\,(v_{\phi o}B_{ro} - v_{ro}B_{\phi o})$$

$$P_{ev12} = E_{zo}\,(v_{\phi 1}B_{ro} - v_{r1}B_{\phi o}) \quad (4.3.6)$$

The equations defining P_{v11}, L_{v11}, P_{ev11}, represent the power conversion due to the interaction of the unperturbed velocity field with the first order electromagnetic field components. The remaining three equations defining P_{v12}, L_{v12}, and P_{ev12} correspond to the interaction of the perturbation velocity field with the unperturbed electromagnetic fields.

Since the above expressions represent power densities, the total power converted in the generator chamber is given by the corresponding volume integrals:

$$P_{1i} = \int_0^L \int_0^{2\pi} \int_{\rho_o}^{\rho_I} P_{v1i}\, r\, dr\, d\phi\, dz \quad ;$$

$$L_{1i} = \int_0^L \int_0^{2\pi} \int_{\rho_o}^{\rho_I} L_{v1i}\, r\, dr\, d\phi\, dz \quad ;$$

$$P_{1ei} = \int_0^L \int_0^{2\pi} \int_{\rho_o}^{\rho_I} P_{ev1i}\, r\, dr\, d\phi\, dz$$

$$i = 1,\ 2 \tag{4.3.7}$$

In fact the three above quantities are not independent since

$$P_{1ei} = P_{1i} - L_{1i} \quad ; \quad i = 1,\ 2 \tag{4.3.8}$$

Thus calculating any two of the above integrals the third quantity can be computed using equation (4.3.8).

Performing the integrations indicated the following expressions are obtained:

$$\begin{aligned}
P_{11} = \frac{4\pi}{m}\sqrt{\frac{2}{\pi}} \sum_{s=1}^{\infty} \Bigg\{ & \left(\frac{1}{s^2 mr}\right) \Bigg[\left(\left(\frac{Q^2 - \rho_I{}^2 V_I{}^2}{2}\right) C_s - \right. \\
& \left. - \rho_I V_I Q A_s\right) I_1(smr) + \left(\left(\frac{Q^2 - \rho_I{}^2 V_I{}^2}{2}\right) D_s - \rho_I V_I Q B_s\right) K_1(smr)\Bigg] \\
& - \frac{\omega}{s^3 m^2} \Big[(QA_s + \rho_I V_I C_s)\, I_o(smr) - (QB_s + \rho_I V_I D_s)\, K_o(smr)\Big] \\
& + \frac{\omega Q r}{2s^2 m}\,(A_s I_1(smr) + B_s K_1(smr)) - \frac{Q^2 + \rho_I{}^2 V_I{}^2}{s^4 m^2}\sqrt{\frac{2}{\pi}}\,\frac{Q}{r^2} + \\
& + \frac{1}{2s}\,(Q^2 + \rho_I{}^2 V_I{}^2)\, C_s \int r^{-1} I_o(smr)\, dr \\
& - \frac{1}{2s}\,(Q^2 + \rho_I{}^2 V_I{}^2)\, D_s \int r^{-1} K_o(smr)\, dr \Bigg\}_{\rho_o}^{\rho_I}
\end{aligned} \quad ; \tag{4.3.9}$$

The two integrals $\int r^{-1} I_o(smr)\,dr$ and $\int r^{-1} K_o(smr)\,dr$ indicated in the above expression can be evaluated numerically, using the series-definition of the modified Bessel functions, for example.

$$
\begin{aligned}
P_{ell} = \frac{4\pi\omega}{m^2}\sqrt{\frac{2}{\pi}} \sum_{s=1}^{\infty} \Big\{ & \frac{\omega Q}{s^4 m}\sqrt{\frac{2}{\pi}}\, r^2 + \\
& + \frac{1}{s^3 m}\Big[(QA_s + \rho_I V_I C_s)\, I_o\,(smr) - \\
& - (QB_s + \rho_I V_I D_s)\, K_o\,(smr)\Big] - \frac{\omega^2 r^2}{s^3 m}\Big[C_s(I_o(smr) - \\
& - \frac{2I_1(smr)}{smr}) - D_s\,(K_o(smr) + \frac{2K_1(smr)}{smr})\Big] - \\
& - \frac{Qr}{2s^2}\,(A_s I_1(smr) + B_s K_1(smr))\Big\}_{\rho_o}^{\rho_I}
\end{aligned}
\qquad (4.3.10)
$$

Furthermore, the integrals expressing the interaction of the perturbation velocity field with the unperturbed electromagnetic field can be written as follows:

$$
\begin{aligned}
P_{12} = \; & \frac{\pi}{2}\,\omega S\Big[r^2(\psi_2 - \frac{\omega}{2S}\, r^2)\Big]_{\rho_o}^{\rho_I} + \\
& + \pi\, \rho_I V_I S\,\Big[\frac{\omega}{S}\, r^2 - 2\psi_3\Big]_{\rho_o}^{\rho_I} - \\
& - \pi S\Big[(Q\psi_1 + \rho_I V_I \psi_2)\Big|_{\rho_o}^{\rho_I} + \\
& + 2\int_{\rho_o}^{\rho_I} (Q\,\frac{\psi_1}{r} + \rho_I V_I\,\frac{\psi_2}{r}\,)\,dr\,\Big]
\end{aligned}
\qquad (4.3.11)
$$

where the first order stream functions ψ_1, ψ_2 and ψ_3 are given by equations (3.1.37) to (3.1.39).

Finally, the net or effective power generated by the interaction of the perturbation velocity with the unperturbed electromagnetic fields is given as

$$P_{el2} = \pi\omega S \left[\frac{r^2}{4}\left(\frac{\omega}{S} r^2 - 2\psi_2\right)\Big|_{\rho_o}^{\rho_I} - \int_{\rho_o}^{\rho_I} r^2 \psi_3' \, dr\right] \quad (4.3.12)$$

V. RESULTS AND CONCLUSIONS

1. Numerical Computations

The following dimensionless quantities have been computed for a number of basic hydrodynamic and geometric input parameters:

a) the zeroth order average power density $\overline{P}_o$

b) the zeroth order average ohmic loss density, $\overline{L}_o$

c) the zeroth order average net (i.e. effective) power density, $\overline{P}_{eo}$

d) the first order average net power density $\overline{P}_{e1}$

e) the electrical efficiency of the power converter system based on zeroth order quantities η.

The average power densities described above are defined as power produced in the generator chamber divided by the volume of the chamber:

$$\left. \begin{aligned} \overline{P} &= \frac{P}{\text{vol.}} \;; \quad \text{where} \\ P &= \int_V P_v \, dv \\ \text{vol.} &= \int_V dv \end{aligned} \right| \qquad (5.1.1)$$

The average power densities (i.e. average power generated per unit volume of the generator chamber) give a convenient basis for comparison of the performance data obtained for different geometric configurations and input parameters.

Since

$$
\left.
\begin{aligned}
P_v &= \frac{(P_v)\ \text{dimensional}}{\sigma(\Delta V)^2\ B_o{}^2} \quad \text{and} \\[2ex]
\text{Vol.} &= \frac{(\text{Vol.})\ \text{dimensional}}{(\Delta R)^3}
\end{aligned}
\right| \qquad (5.1.2)
$$

the dimensional average power density can be obtained for each particular application by computing the following quantity:

$$
\left(\frac{P}{\text{vol.}}\right)_{\text{dimensional}} = \overline{P}\left(\frac{\sigma(\Delta V)^2 B_o{}^2}{(\Delta R)^3}\right) \qquad (5.1.3)
$$

where $\overline{P}$ is defined by equation (5.1.1) and is computed for various sets of input parameters. The numerical results are tabulated below.

The efficiency computations are based on the zeroth order quantities. The first order net power has been also computed, its influence on the zeroth order quantities is, however, negligible.

The following dimensionless quantities and ratios were chosen as basic parameters (i.e. input data for the subsequent calculations:

the injection velocity: $(V_I)_{\text{non dim.}}$

the ratio of the exit velocity and the injection velocity: V_o/V_I

the radius ratio of the inside and the outside cylinders:	R_o/R_I
the ratio of the length of the annulus and the radius of the outside cylinder:	L/R_I
the angular velocity of the magnetic field:	ω

Introducing the notation

$$\Omega \equiv \frac{V_I}{\omega R_I} \; ; \; \Lambda \equiv \frac{-V_o}{V_I} \; ; \; L_I \equiv \frac{L}{R_I} \tag{5.1.4}$$

the above quantities can be written as

$$\left.\begin{aligned}
(V_I)_{\text{non dim.}} &= \frac{\Omega}{\Omega - 1} \; ; \\
(V_o)_{\text{non dim.}} &= \frac{\Lambda\Omega}{\Omega - 1} \; ; \\
(\omega)_{\text{non dim.}} &= \frac{1 - \beta}{\Omega - 1} \; ; \\
(R_I)_{\text{non dim.}} &= \frac{1}{1 - \beta} \; ; \\
(R_o)_{\text{non dim.}} &= \frac{\beta}{1 - \beta} \; ;
\end{aligned}\right| \tag{5.1.5}$$

where $\beta = R_o/R_I$ as has been defined previously.

The length ratios L_I, enters in the definition of the parameter "m" applied throughout the finite Fourier transforms and it can be written now as

$$m = \frac{\pi(1 - \beta)}{L_I} \qquad (5.1.6)$$

The parameter Ω is closely related to the slip factor s, defined in the previous section. In fact, for the slip factor s, measured at the outside radius R_I the following relation can be derived:

$$\Omega = 1 - s \; ; \quad \text{or} \quad s = 1 - \Omega \qquad (5.1.7)$$

The zero slip condition corresponds to $\Omega = 1$. One may note, however, even if the $\Omega = 1$ condition is satisfied at the outer radius, for the present configuration the bulk of the fluid will still have an excess velocity over the propagating magnetic field due to the vortex type velocity distribution given as $v_{\phi o} = (\rho_I V_I/r)$. The larger the duct width, $\Delta R = R_I - R_o$, is, the farther is the bulk of the fluid from the no-slip condition, even if $\Omega = 1$.

Throughout all computation the magnitude of the exit velocity was assumed to be 1% of that of the injection velocity which corresponds to $\Lambda = -.01$.

The choice of Ω is such that for reasonable generator sizes and commercially available frequencies (60 cycles per sec., for example) the injection velocity would have a value at which the fluid can be still treated as an incompressible medium. This condition is satisfied for $\Omega = 1.5$.

The length ratio L_I varies between the limits 1.0 and 4.0 which correspond to reasonable design configurations. For one set of input data the length ratio $L_I = 50.0$ was chosen to obtain information about the performance characteristics corresponding to reduced end-plate effects.

The magnetic pressure coefficient S is assumed to equal to unity.

For a laboratory size generator ($R_I = 0.5$ m to 1.0 m) operating on a working medium whose electrical conductivity does not exceed 100 mho/m the above assumptions restrict the values of the magnetic Reynolds number so that $Rm < .01$. Since the nondimensional first order power output was found to be of the same order of magnitude as the corresponding zeroth order power, its effect on the performance characteristics is negligible.

A "Burroughs 220" digital computer was used for the performance calculations.

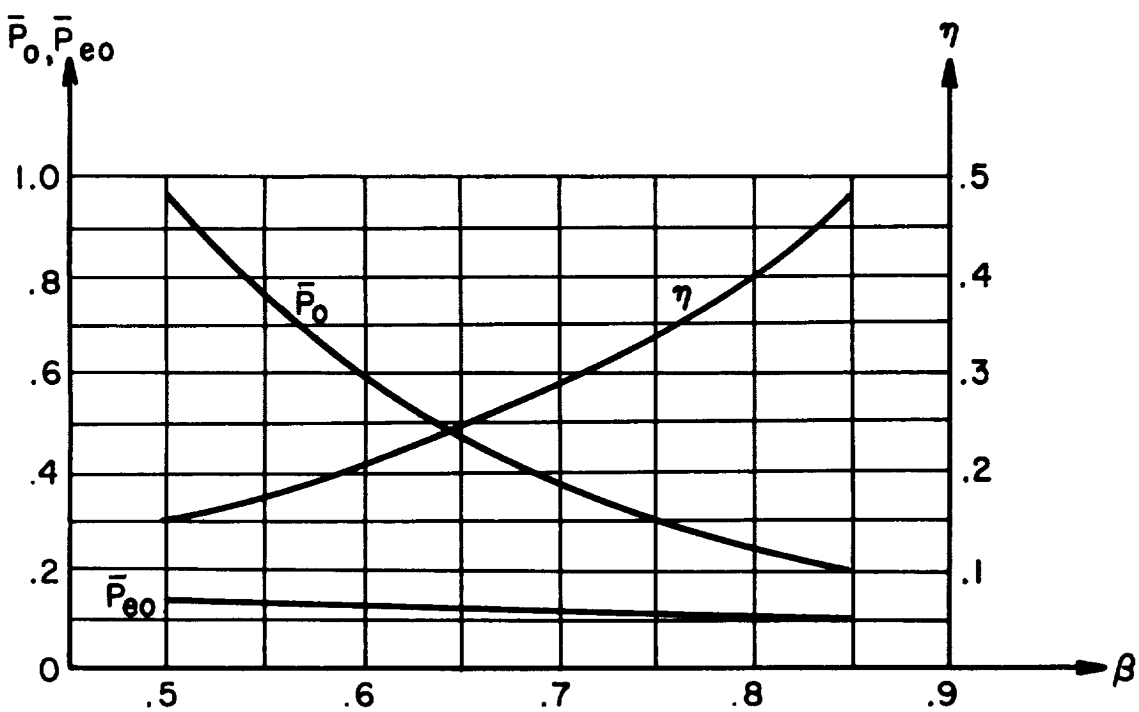
The numerical results are tabulated as follows:

TABLE NO. 1

$\Omega = 1.5 \qquad L_I = 1.0$

β	$\bar{P}_o$ (zeroth order)	$\bar{P}_{eo}$ (zeroth order)	η
.5	.9634	.1455	.1510
.6	.6259	.1332	.2123
.7	.4082	.1222	.2994
.75	.3291	.1165	.3538
.8	.2651	.1102	.4156
.85	.2135	.1031	.4830

$\bar{P}_o$, $\bar{P}_{eo}$ η



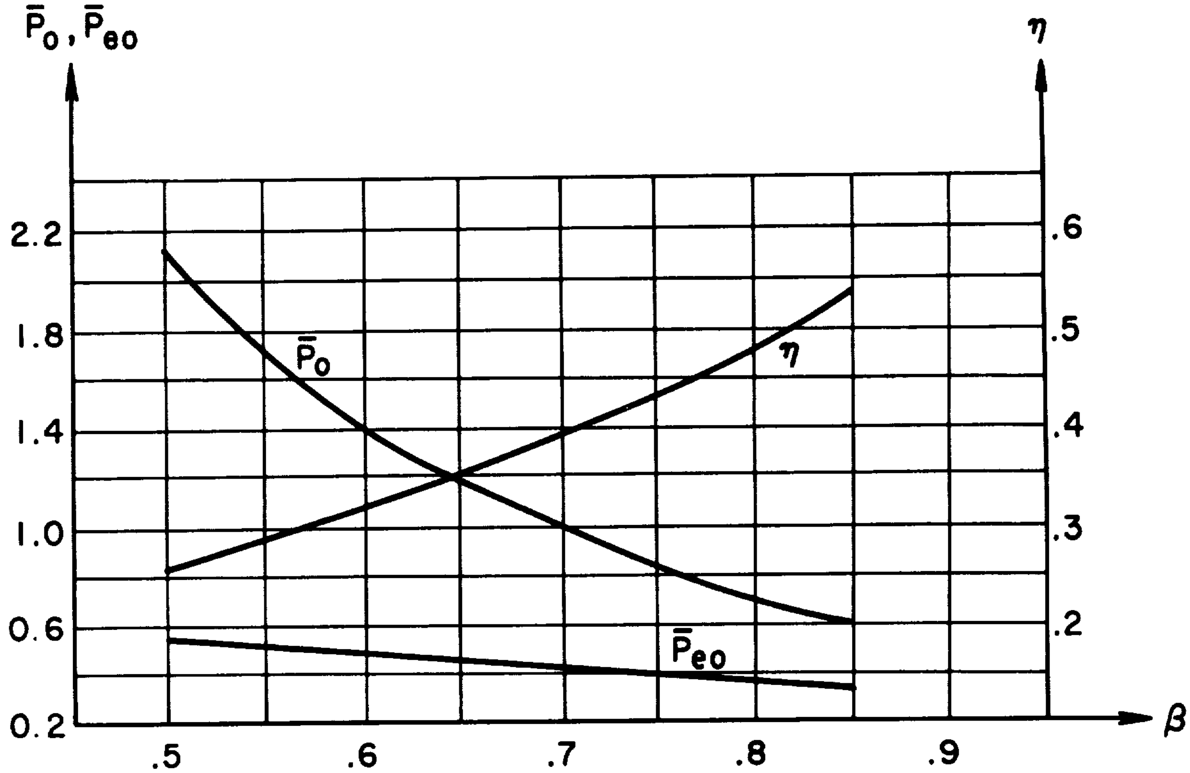
The plot of the average power density $\bar{P}_o$ and average effective power density $\bar{P}_{eo}$ vs. the radius ratio β for

$\Omega = 1.5 \qquad L_I = 1.0$

TABLE NO. 2

$\Omega = 1.5 \qquad L_I = 2.0$

β	$\bar{P}_o$ (zeroth order)	$\bar{P}_{eo}$ (zeroth order)	η
.5	2.1204	.5495	.2591
.6	1.4994	.4863	.3273
.7	1.0466	.4251	.4061
.75	.8738	.3945	.4493
.8	.7365	.3640	.4943
.85	.6170	.3333	.5403



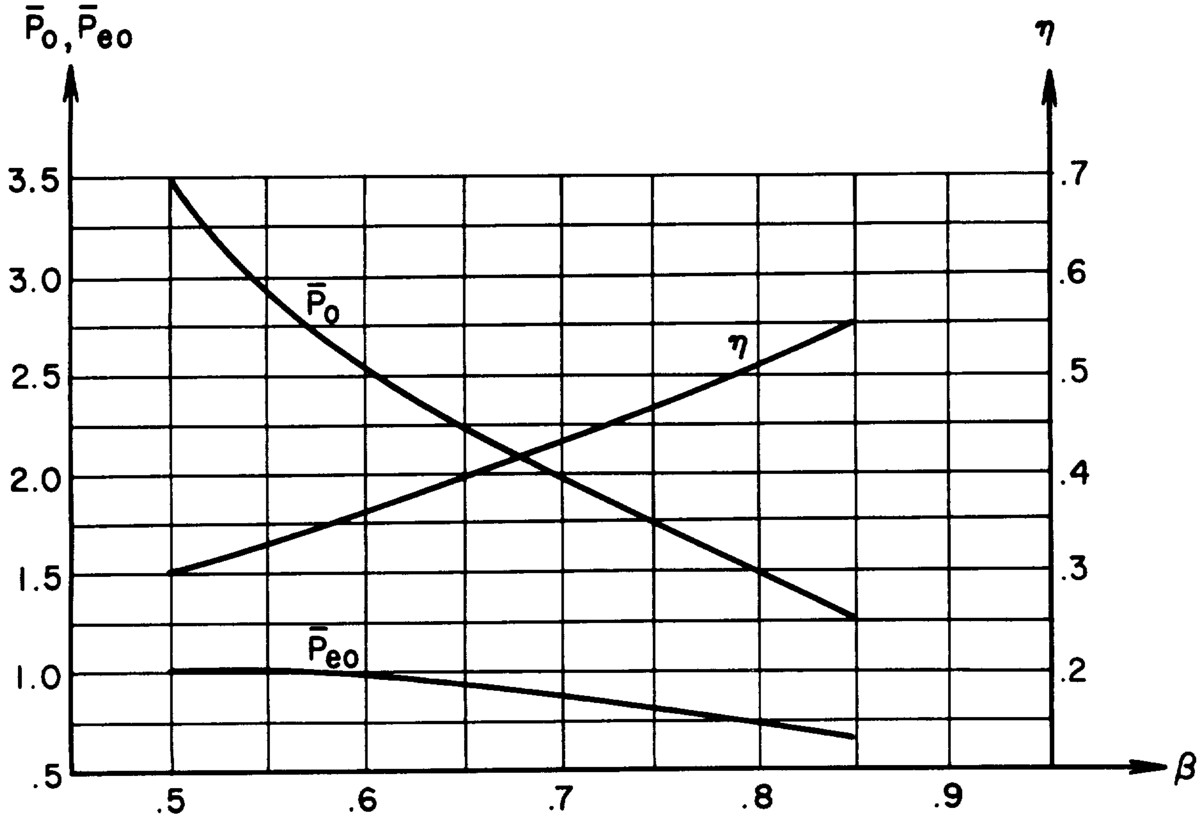
The plot of the average power density $\bar{P}_o$ and average effective power density $\bar{P}_{eo}$ vs. the radius ratio β for

$\Omega = 1.5 \qquad L_I = 2.0$

TABLE NO. 3

$\Omega = 1.5 \qquad L_I = 4.0$

β	$\bar{P}$ (zer8th order)	$\bar{P}_{eo}$ (zer8th order)	η
.5	3.5045	1.0688	.3050
.6	2.6173	.9705	.3708
.7	1.9591	.8600	.4420
.75	1.6927	.8114	.4794
.8	1.4595	.7554	.5175
.85	1.2554	.6980	.5560



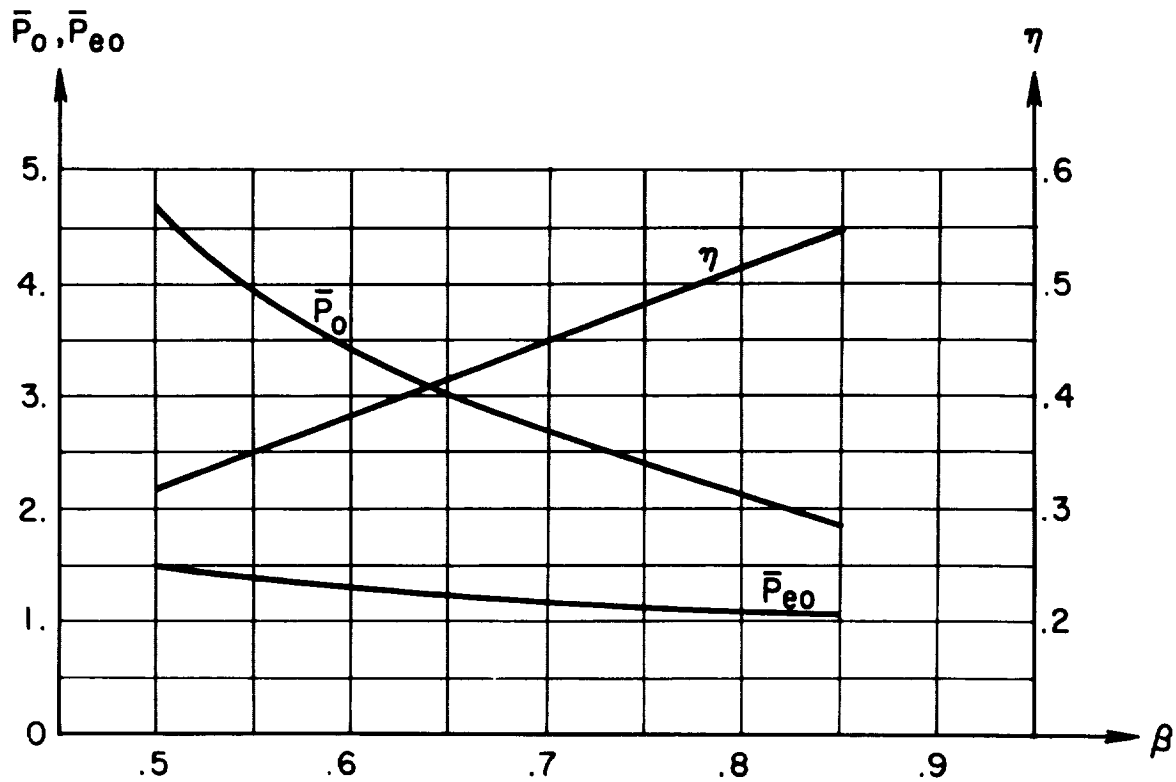
The plot of the average power density $\bar{P}_o$ and average effective power density $\bar{P}_{eo}$ vs. the radius ratio β for

$\Omega = 1.5 \qquad L_I = 4.0$

TABLE NO. 4

$\Omega = 1.5 \qquad L_I = 10.0$

β	$\bar{P}_o$ (zeroth order)	$\bar{P}_{eo}$ (zeroth order)	η
.5	4.5701	1.4734	.3224
.6	3.5388	1.3671	.3863
.7	2.7445	1.2461	.4540
.75	2.4128	1.1803	.4892
.8	2.1129	1.1093	.5250
.85	1.8577	1.0365	.5609

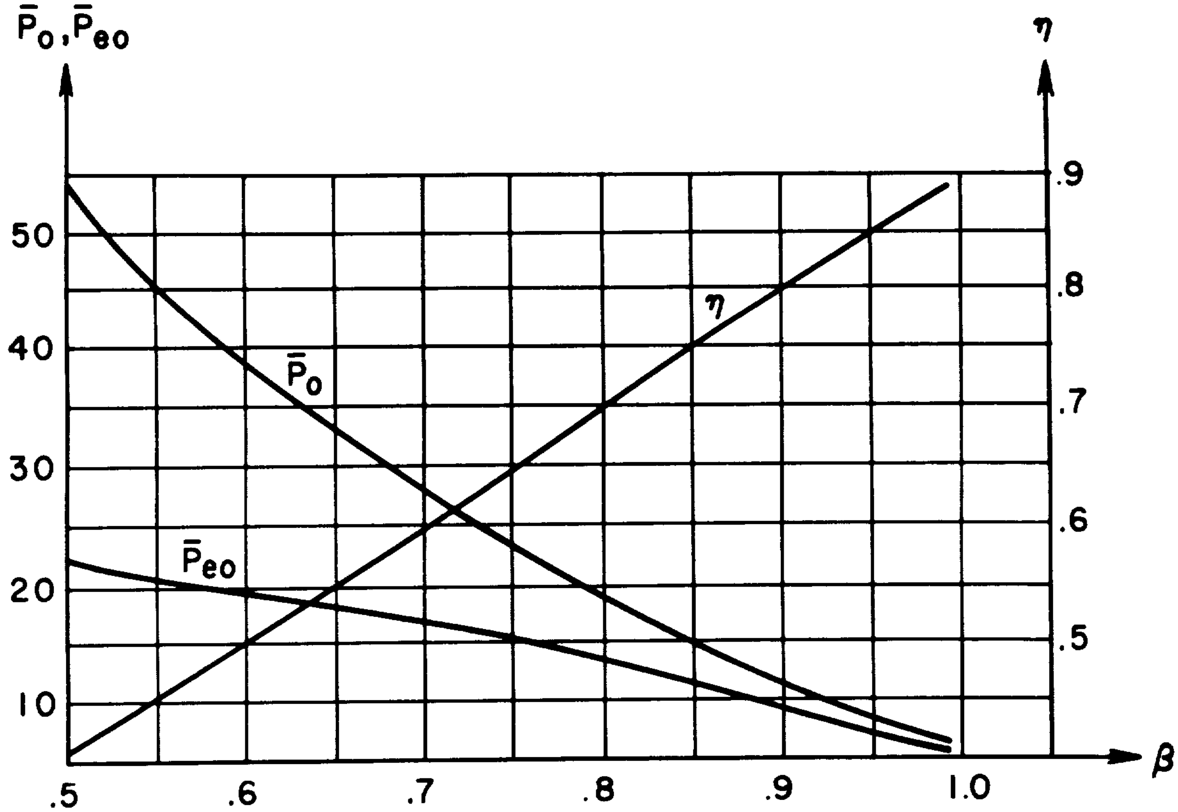


The plot of the average power density $\bar{P}_o$ and average effective power density $\bar{P}_{eo}$ vs. the radius ratio β for

$\Omega = 1.5 \qquad L_I = 10.0$

TABLE NO. 5 $\Omega = 1.1$ $L_I = 50.0$

β	$\bar{P}_o$	$\bar{P}_{eo}$	η
.5	54.437	22.65	.4161
.6	39.69	19.97	.5033
.7	28.06	16.78	.5979
.75	23.39	15.12	.6466
.8	18.94	13.22	.6980
.85	14.97	11.26	.7517
.9	11.35	9.14	.8058
.99	5.95	5.19	.8769

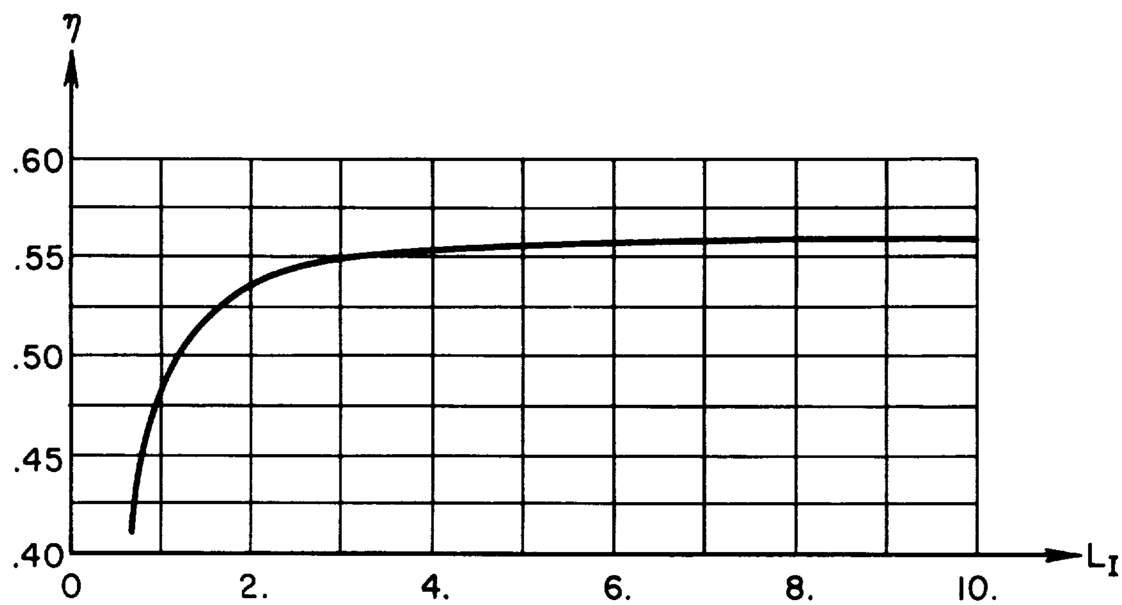


The plot of the average power density $\bar{P}_o$ and average effective power density $\bar{P}_{eo}$ vs. the radius ratio β for

$\Omega = 1.1$ $L_I = 50.0$

TABLE NO. 6

	L_I	η
β = .85	1.0	.4830
	2.0	.5403
Ω = 1.5	4.0	.5560
	10.0	.5609



The plot of the efficiency η vs. the length ratio L_I for

β = .85 Ω = 1.5

2. Discussion of the Results.

The numerical data tabulated in the previous section are indicative of the performance characteristics of a vortex type MHD induction generator and can also be used for comparing the system discussed here with a linear induction generator, analyzed by Bernstein (ref. [12]).

An inspection of the tabulated results reveals the following:

As the width of the duct decreases (i.e. β increases) the average power generated per unit volume decreases, but at the same time the efficiency of the conversion cycle increases. Thus the effective power decreases at a much smaller rate than the total power. Both the total and the effective power decrease quite rapidly at small values of β, the slope of the power curves decreases as $\beta \to 1$.

Furthermore, with the increase of the length of the annulus (i.e. of the L_I ratio) both the power output per unit volume (including the total and the effective powers) and the cycle efficiency increase. The improvement of the performance characteristics corresponding to increasing L_I ratio is more pronounced at small values of L_I ($L_I \simeq O(1)$). For $1 \ll L_I$ the efficiency changes very slowly as can be seen from Table No. 6.

The changes described above correspond to a given ratio of injection to phase velocity, Ω.

The decrease of the power generated per unit volume and the increase of the efficiency corresponding to increasing β can be explained as follows: as the width of the duct decreases the largest value of the tangential velocity ($v_\phi = \rho_I V_I/r$) is reduced for a given (constant) injection velocity.

Thus the bulk of the fluid moves with a reduced average velocity. For a given Ω, where $1 < \Omega$ (that is, given an injection velocity exceeding the phase velocity) the reduction of the average fluid velocity reduces the actual slip between the bulk of the fluid and the propagating magnetic field.

Equation (4.1.9a) indicates (see also ref. [12]), that the smaller the slip is the larger the efficiency of the energy-conversion process. Zero slip corresponds to maximum efficiency and minimum (i.e. zero) power production, the latter being caused by the absence of intersection of the magnetic flux lines by the moving conductor. Hence, the reduction of the average fluid velocity, corresponding to large values of β, causes an increase of the conversion efficiency due to the reduced slip factor on one hand, and it reduces the power generated per unit volume due to the decreased interaction between the velocity and electromagnetic fields on the other hand.

Furthermore, the above argument indicates that for a given value of the duct width (i.e. fixed β ratio) the power conversion density can be increased by increasing the injection to phase velocity ratio: Ω.

The general improvement of the operational characteristics with the increase of the annulus length is explained by the fact that the relative magnitude of the eddy (or closing) currents is reduced as the length of the cylinders increases. The end plate effects are more pronounced for small values of the length ratio L_I ($L_I \leq 3.0$) and they become practically negligible when the ratio $L_I = 6.0$ is exceeded.

At this point comparison between the linear generator described in reference [12] by Bernstein and the limiting case of the vortex generator configuration discussed here can be made.

Bernstein analyzed the motion of a conducting medium moving in the x direction with a given, uniform velocity V_x between two infinite, parallel planes placed at $y = \pm a$ and its interaction with a traveling magnetic field given by $B_y \simeq \text{const.} [\exp (kx - \omega t)]$ propagating in the x direction with a phase velocity equal to ω/k. The induced currents are directed along the z-axis and their path is not restricted by any nonconducting end plates.

The present analysis differs in two significant aspects from Bernstein's in that it is based upon a finite configuration and a realistic velocity distribution. The currents are deflected from their principal direction by nonconducting end plates, placed at a finite distance from each other.

In the limiting case, however, when the vortex velocity distribution approaches a uniform velocity (that is, when the width of the duct is reduced to near zero) and the end plates of the cylindrical annulus are separated by a considerable distance (the length of the cylinder approaches to infinity); the characteristics of the vortex generator should converge to those of a linear generator.

Indeed, when $L_I \rightarrow \infty$, $\beta \rightarrow 1.0$ and $\Omega \rightarrow 1.0$, the efficiency of the conversion cycle should approach its limiting value: $\eta \rightarrow 1.0$.

In order to check the consistency of the numerical computations presented here, a set of input parameters were chosen so that the asymptotic value of the efficiency could be approached.

For $L_I = 50.0$, $\beta = .99$ and $\Omega = 1.1$ the electrical efficiency computed is .8769 (see Table 5). For the corresponding linear generator with $s = -.1$ ($s = 1 - \Omega$) the efficiency can be computed using the expression given by Bernstein:

$\Omega = \dfrac{1}{1 - s} = .909$. The difference between the two efficiencies is due to the end plate effects. Although the $L_I = 50.0$ ratio reduces the relative importance of the closing currents substantially, it does not eliminate them completely.

Finally, the question as to which of the configurations (linear or vortex) is better, cannot be answered until the characteristics of a finite linear generator are known.

3. Summary

The primary purpose of the present analysis was to gain information about the performance characteristics of a vortex type magnetohydrodynamic AC generator. The configuration chosen for the analysis was suggested by its compactness and the relatively small amount of work done so far in the field of rotating magnetohydrodynamic flows with emphasis on power conversion. The analysis has been carried out by taking advantage of perturbation technique and other approximate methods used in operational mathematics. The numerical computations were carried out on an electronic digital computer; the obtained results are consistent with data, published by other authors. In particular, for high length-radius ratios, the generator characteristics converge to those of a linear generator.

For a given injection to phase velocity ratio Ω, the power output is largest at small radius ratios β, but the operation is the least efficient there. As β increases the efficiency increases, but the power output decreases. Thus the choice of design parameters for a practical generator must be based on a compromise between efficient operation and maximum power output. The range $.75 \leq \beta$ seems to be best for practical applications.

Since the improvement of the efficiency with increasing the length ratio L_I is quite slow after the ratio $L_I \simeq 6.0$ is exceeded, the practical design values should be chosen in the interval $3.0 \leq L_I \leq 6.0$.

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LIST OF SYMBOLS

$a_{1s}, b_{1s}, c_{1s}, d_{1s}$	constants, defined by Eq. (3.2.40)
$a_{2s}, b_{2s}, c_{2s}, d_{2s}$	constants, defined by Eq. (3.2.41)
$a_{3s}, b_{3s}, c_{3s}, d_{3s}$	constants, defined by Eq. (3.2.41)
A_s, B_s, C_s, D_s	constants, defined by Eqs. (2.2.39), (2.2.40)
$A_{11}, A_{21}, A_{31}, B_{11}, B_{21}, B_{31}, C_{21}, C_{22}, C_{32}$	constants defined by Eqs. (3.1.43)
A_o	total area of exit ports (m^2)
$\vec{B}$	magnetic induction field (volt sec/m^2)
C_{ij}	function obtained through a finite Fourier cosine transform
D	constant defined by Eq. (2.1.11)
$\vec{E}$	electrical field intensity (volt/m)
$f_n(r)$	function defined by Eq. (3.1.28)
$f_i(r,s,z);\ i = 1,2$	functions defined by Eq. (3.2.11)
$\vec{F}$	body force (Kg)
F	function defined by Eq. (3.1.17)
$F_{1s}^{(r)},\ F_{2s}^{(r)}$	functions defined by Eqs. (4.2.9)
$F_{1s}^{(\phi)},\ F_{2s}^{(\phi)}$	functions defined by Eqs. (4.2.10)
$F_{1s}^{(z)},\ F_{2s}^{(z)}$	functions defined by Eqs. (4.2.11)
$F_1^{(r)}, F_2^{(r)}, F_3^{(r)}$	functions defined by Eq. (3.1.19)
$g(r)$	function defined by Eq. (3.1.24)
$\vec{H}$	magnetic field intensity (amp/m)

$\vec{I}$	current density (amp/m^2)
$\vec{J}_o$	current density in field coils (amp/m^2)
K	thermal conductivity $(Cal/m.sec.^{o}K)$
$\vec{K}$	surface current density (amp/m^2)
L	length of the generator chamber (m)
L_V	ohmic losses per unit volume (dimensionless)
$\bar{L}$	average ohmic losses per unit volume (dimensionless)
L_I	length ratio $(= L/R_I)$
LHS	left hand side
m	constant $(= \pi/L^*)$
M	constant defined by Eq. (3.1.17)
MHD	"magnetohydrodynamic"
N	magnetic interaction parameter
p	pressure (Kg/m^2)
p_I	injection pressure (Kg/m^2)
p_T	total pressure (Kg/m^2)
P_v	power per unit volume (dimensionless)
P_{ev}	effective power per unit volume (dimensionless)
$\bar{P}$	average power per unit volume (dimensionless)
$\bar{P}_e$	average net power per unit volume (dimensionless)
Q	constant related to the total volume flow
r	radius (m)
r_I	radius ratio defined by Eq. (2.1.19)
r_1	radius $(= mr)$

R_o	inside radius (m)
R_I	outside radius (m)
Re	hydrodynamic Reynolds number
Rm	magnetic Reynolds number
RHS	"right hand side"
s	slip factor defined by Eq. (4.1.9)
S	magnetic pressure coefficient
S_{ij}	function obtained through a finite Fourier sine transform
t	time (sec.)
T	temperature (^oK)
V	velocity (m/sec.)
V_I	injection velocity (m/sec.)
V_o	exit velocity (m/sec.)
z	coordinate (m)
z_1	coordinate (= mz)
$Z_\lambda(r)$	cylindrical function
ΔR	duct width ($= R_I - R_O$) (m)
ΔV	velocity differential ($= V_I - \omega R_I$) (m)
β	radius ratio ($= R_O/R_I$)
δ	ratio defined by Eq. (3.1.17)
ε	electric permittivity ($coul^2/Kg.m^2$)
η	electrical efficiency
λ	index for Bessel functions
μ	magnetic permeability (volt. sec/amp.m)
ν	kinetic viscosity (m^2/sec.)

ρ	hydrodynamic density (kgm/m^3)
ρ_e	electrical charge density $(cou\,\ell/m^3)$
ρ_o	dimensionless radius $(= R_O/\Delta R)$
ρ_I	dimensionless radius $(= R_I/\Delta R)$
σ	electrical conductivity (amp/volt m)
ϕ	azimuthal coordinate
ϕ_o	potential function describing the zeroth order electrostatic field
Φ	hydrodynamic dissipation function
ψ	stream function for first order velocity
ω	angular velocity (1/sec)

Subscripts

x,y,z r,ϕ,z	components along the corresponding coordinate axes
0	zeroth order
1	first order
c	complementary
n	normal
p	particular
t	tangential

Superscripts

(')	quantity expressed in a rotating coordinate system; or derivative with respect to γ
*	nondimensional quantity